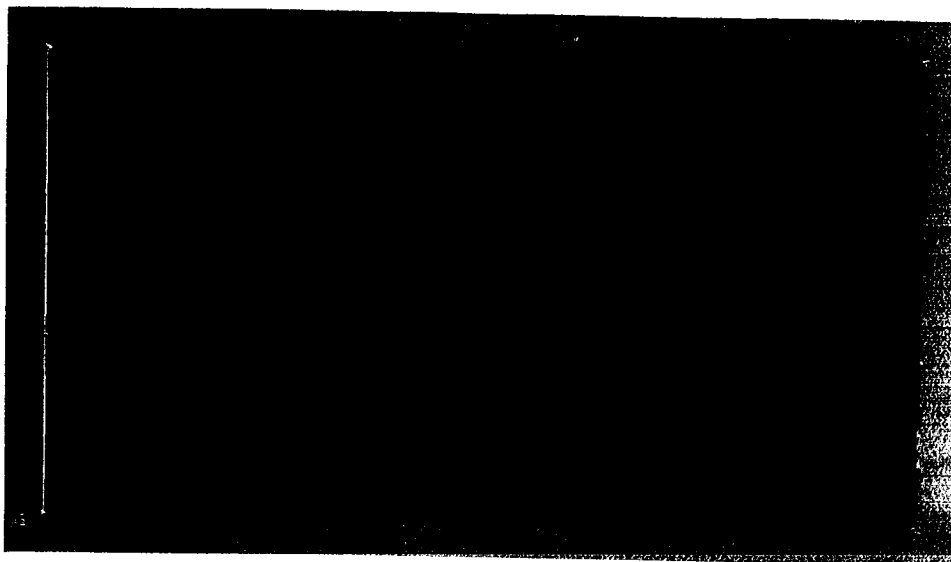


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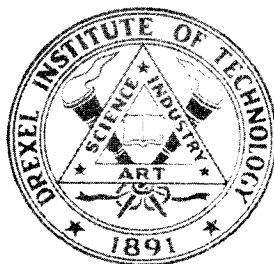


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METHODS FOR SYSTEMATIC
GENERATION OF
LIAPUNOV FUNCTIONS (PART ONE)

Submitted by

R. L. Drake

Mathematics Department

CONTRACT NO. NAS 8-11196

George C. Marshall Space Flight Center
National Aeronautics and Space Administration
Huntsville, Alabama

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(ii)

ABSTRACT

This report summarizes much of the work that has been done in the field of stability theory with regards to the generation of Liapunov functions. The emphasis of the report has been to survey and discuss the work of American engineers and mathematicians in this area. But since most of the work was motivated by Russian mathematicians and engineers, this report also includes a sizable discussion of the Russian contributions. Reference is also made to the contributions due to mathematicians in England, Japan and Italy. Under separate cover, the writers of this report submit a sizable list of references in the stability field and a summary of the theorems and definitions which are important in the analysis of stability problems.

(iii)

LIST OF SYMBOLS

(most symbols are defined where they are used in the report and will not be repeated here)

V = usually denotes a scalar function, or functional which is a Liapunov function or a candidate for a Liapunov function.

$\| \underline{x} \|$ = usually denotes the Euclidean norm of an n -dimensional vector, defined as:

$$\left[x_1^2 + \dots + x_n^2 \right]^{1/2}.$$

$t \in [a, b]$ means $a \leq t \leq b$.

$t \in (a, b]$ means $a < t \leq b$.

$t \in (a, b)$ means $a < t < b$.

$a \in A$ means that element a is a member of set A .

\underline{A}_T = transpose of matrix \underline{A}

\underline{A}^* = conjugate transpose of matrix \underline{A} .

$\dot{\underline{x}}$ = time derivative of the vector function, $\underline{x} \equiv \underline{x}(t)$.

E_n = Euclidean n -space.

C^n = The class of functions having continuous n -th order partial derivatives.

$\underline{\nabla} V$ = gradient of the scalar function V .

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SUMMARY

In this part of the report certain applications of Liapunov's stability theory to the stability analysis of nonlinear control systems are reviewed. Section 1 gives some definitions and basic theorems and Section 2 is a discussion of the equations describing the control system to be considered here.

The problem of Lur'e is treated in Sections 3 and 4 and the results of Lur'e are presented as well as more recent results obtained by Lefshetz. Following Section 4 are three examples. Section 5 modifies the restrictions on the admissible nonlinear characteristics and more useful results are obtained than in the preceding two sections. The treatment of Aizerman and Gantmacher is followed here. An example follows.

In Section 6 the problem of Aizerman is considered. A treatment of the general second order case due to Krasovskii is presented. An example demonstrates the verification of Aizerman's conjecture for a particular third order control system.

Two theorems of Popov along with a discussion of the Liapunov function of Popov are presented in Section 7, 8 and 9. An important theorem due Kalman relating the Popov criterion with criterion obtained using Liapunov theory with a Popov type Liapunov function is given in Section 10.

There follows a compendium of examples illustrating applications of the theory to particular problems as well as the derivation of some simplified criteria from the more general results.

SECTION ONE

INTRODUCTION

Prepared by: R. L. Drake

INTRODUCTION

This report deals with new methods of generating Liapunov functions. The attempt was made to survey the stability field in order to summarize all possible methods of obtaining Liapunov functions. Because of the great scope of current usage of Liapunov theory, or generalizations of Liapunov theory in the fields of linear and nonlinear ordinary differential equations (both autonomous and non-autonomous), differential-differences equations, functional equations, stochastic differential equations and others, our objective of a complete and thorough survey of the field was not accomplished. But we do feel that much has been accomplished, these accomplishments will now be outlined in the following paragraphs.

Section Two of this report is concerned with Liapunov functions constructed from the various first integrals which occur in certain dynamic systems. This use of first integrals was one of the motivating factors in Liapunov's original work; and we feel that this area of Liapunov theory is still one of the most important sources of usable Liapunov functions.

Section Three discusses the work of Puri and his colleagues at the University of Pennsylvania; this work is a "nonlinear analogue" of the Linear Theory developed by Routh and Hurwitz.

Section Four considers a method of generating Liapunov functions which many times is the first technique that engineers attempt. The work which we report here deals with some of the more important results which have been obtained.

Section Five deals with the variable gradient method and the many modifications of the procedure. This method is very useful but has a built-in trial-and-error procedure which may prove extremely difficult in its application to certain systems.

Section Six considers the use of Liapunov functions in analyzing the stability of automatic control systems. This work originated in the Russian school with

investigators such as Lur'e, Malkin and Popov; the major American contributors are S. Lefschetz and J. LaSalle.

Section Seven discusses the work of many contributors, with the major motivation coming from some early theorems of Krasovskii.

Section Eight deals with the partial differential equations of Zubov and the extension of this work due to Szegö.

Section Nine considers the allied topics of boundedness and differential inequalities. This work is due mainly to Bellman and Yoshizawa(Japan).

Section Ten presents some of the "Liapunov theory" results obtained for non-autonomous systems. These results are mainly for linear systems, with some treatment of the nonlinear problem. We might add, that much of the work in Section Nine has application to the time-varying stability problem.

Section Eleven is a miscellaneous section. It contains some very important results of Leighton and Skidmore for autonomous systems. It contains reference lists dealing with stochastic stability, functional-differential equations and topological dynamics.

Under separate cover we will submit a reference list on stability theory and a compendium of theorems and definitions. Also, any general recommendations and observations will be given by the principal investigator at a contractor's meeting at Huntsville, Alabama.

A note about the "physical structure" of this report is in order. Each section will have its pages numbered independent of the other sections; this will also be true of theorems, definitions and equations. The symbols will be defined as used and no complete list of symbols is given. No list of illustrations is given; the few figures which occur, occur at the location in the report at which they are being discussed.

One other note deals with the method of "Separation of Variable" for generating Liapunov functions. This method was mentioned by J. P. LaSalle in a speech given in Iowa in 1964. During the contract period, the principal investigator could not find any further information on this topic. In what follows, we give a very rough sketch of the method:

We attempt to construct the following type of V-function for an n-dimensional system,

$$V = V_1(x_1) + V_2(x_2) + \dots + V_n(x_n).$$

The class of systems covered is

$$\dot{x}_i = \sum_{k=1}^n P_{ik}(\underline{x}) f_k(\sigma_k),$$

$$\sigma_k = \underline{a}_k \cdot \underline{x},$$

$$\text{if } \sigma_k \neq 0 \text{ then } \sigma_k f(\sigma_k) > 0.$$

The candidate for V is

$$V = \left[\sum_{i=1}^n \int_{\sigma_{oi}}^{\sigma_i} f_i(\sigma_i) d\sigma_i \right],$$

where

$$\dot{V} = \underline{f}_T \left\{ \underline{P}_T \underline{A}_T + \underline{A} \underline{P} \right\} \underline{f}.$$

The sufficient conditions for stability are then obtained from V and \dot{V} . (The Russians have used this procedure.)

SECTION TWO

INTEGRAL METHODS

Prepared by: R. L. Drake
T. Gifford

INTEGRAL METHODS

SUMMARY

In this section of the report we deal with one of the most basic methods of generating Liapunov functions, "the integral method". Liapunov's work was the outgrowth of Lagrange's theorem of minimum potential energy and the concepts of total energy of a dynamic system. First, we give a brief summary, along with some simple examples, of the use of first integrals as Liapunov functions.

Next, we discuss the use of Chetaev's linear bundles of first integrals as candidates for Liapunov functions. Along with this discussion, we include Pozharitskii's extension of this idea, which is to find a function of the ~~known~~ first integrals of a system which is positive definite and use this function as a Liapunov function.

Then a short discussion is presented concerning the work of Rumiantsev in extending the second method of Liapunov to the problem of the stability of motion of continuous media with respect to a finite number of parameters, which describe the motion through integral expressions.

Finally, the work of Infante, Walker and Clark is considered. Their method was to obtain a first integral of a "nearby" system and then use this function as a candidate for the Liapunov function of the origin system, whose stability analysis is desired.

A FIRST INTEGRAL USED AS A LIAPUNOV FUNCTION

One of the simplest examples of the application of Liapunov's Stability Theory is a dynamic system which possesses a first integral, $V(\mathbf{x}) = C = \text{constant}$. Since $\dot{V}(\mathbf{x}) = 0$, the function $V(\mathbf{x})$ can be used as a Liapunov function to prove that the origin is stable. The best way to discuss this work is by example; thus, we consider a few pertinent examples in the following text.

Consider the following nonlinear conservative system defined by

$$\ddot{x} = 4x^3 - 4x,$$

or in state variable form,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 4x_1^3 - 4x_1.$$

The singular points, or equilibrium points, are found to be:

$$P_1(0,0), P_2(1,0), P_3(-1,0).$$

By considering the total energy,

$$E = \frac{x_2^2}{2} + \int_0^{x_1} (-4x_1^3 + 4x_1) dx_1 = \frac{x_2^2}{2} - x_1^4 + 2x_1^2$$

where $\dot{E} = 0$, as a Liapunov function, we see that the origin is stable and that the boundary of the stability region passes through P_2 and P_3 and thus is defined by:

$$E = E(x_1, x_2) = E(-1,0) = 1 = \frac{x_2^2}{2} + 2x_1^2 - x_1^4.$$

Another example is a mathematical pendulum defined by, [5] * ,

$$\ddot{x} = -\lambda^2 \sin x,$$

or

$$\dot{x}_1 = -\lambda x_2$$

$$\dot{x}_2 = \lambda \sin x_1.$$

* Numbers in the square brackets refer to references at the end of the section.

The energy integral is given by

$$E = \lambda^2 \left[\frac{x_2^2}{2} \right] + \lambda^2 \int_0^{x_1} \sin x_1 dx_1 = \frac{\lambda^2}{2} \left\{ \left[\frac{x_2^2}{2} \right] + 2 (1 - \cos x_1) \right\},$$

where

$$\dot{E} = \frac{\lambda^2}{2} \left\{ 2x_2 \lambda \sin x_1 + 2(\sin x_1)(-\lambda x_2) \right\} \equiv 0.$$

Therefore, in the neighborhood of (0,0) the system is stable.

A generalization of the previous example is the system defined by, [1],

$$\ddot{x} + g(x) = 0,$$

which describes a unit point mass under a spring force $g(x)$. The state variable formulation is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -g(x_1).$$

Since this is a conservative system, the total energy is

$$E = \frac{x_2^2}{2} + \int_0^{x_1} g(x_1) dx_1,$$

where

$$\dot{E} = x_2(-g(x_1)) + g(x_1)x_2 \equiv 0.$$

Therefore, E is a Liapunov function and proves that (0,0) is a stable equilibrium point if

$$(1) \quad x_1 g(x_1) > 0, \quad x_1 \neq 0,$$

$$(2) \quad g(0) = 0.$$

As a final example we consider a linear constant coefficient dissipative system defined by, [3] ,

$$m\ddot{x} + c\dot{x} + kx = 0,$$

where $c > 0$, $m > 0$ and $k > 0$. The total energy of the system is

$$E = \frac{m}{2}(\dot{x})^2 + \frac{k}{2}x^2,$$

where

$$\dot{E} = m\dot{x} \left(\frac{-c\dot{x} - kx}{m} \right) + kx\dot{x} = -c\dot{x}^2.$$

Thus, by LaSalle's theorem, we have an asymptotically stable region throughout the phase plane.

CHETAEV'S AND POZHARITSKII'S WORK

In reference [25] , Pozharitskii considers the equation of perturbed motion for dynamic systems; namely,

$$\dot{\underline{x}} = \underline{f}(\underline{x}, t),$$

where \underline{x} is an n -vector. These systems admit $p < n$ first integrals

$$U_1(\underline{x}, t), \dots, U_p(\underline{x}, t)$$

which vanish for $\underline{x} = \underline{0}$. If we can now succeed in finding a function

$\phi(U_1, \dots, U_p)$ of known integrals which is definite with respect to \underline{x} ,

then stability of motion is guaranteed by Liapunov Theory. The first theorem concerning the existence of such functions given by Pozharitskii is as follows:

"In order that there exists any definite function $\phi(U_1, \dots, U_p)$, of the known integrals, it is necessary and sufficient that a function

$$\gamma(U_1, \dots, U_p) = U_1^2(\underline{x}, t) + \dots + U_p^2(\underline{x}, t)$$

be definite."

The problem is applying this theorem is still the "old story" of determining whether or not a given function is definite, semi-definite, or indefinite.

Another theorem of Pozharitskii is:

"The function $\chi(U_1, \dots, U_p)$ will be definite only when for at least one of the integrals, say, $U_i(x, t)$, it is possible to find a pair of definite functions $r_i(\|x\|^2)$ and $\rho_i(\|x\|^2)$ such that $U_i^2(x, t) > r_i$ whenever $\|x\|^2 > 0$ and

$$U_1^2 + \dots + U_{i-1}^2 + U_{i+1}^2 + \dots + U_p^2 < \rho_i(\|x\|^2)."$$

From the proof of this theorem it follows that if it is possible to select such a pair of functions for any one of the integrals, then a pair can also be selected for any other integral. This theorem appears to be more useful than the first theorem in setting down guidelines for determining a definite function, $\phi(U_1, \dots, U_p)$.

The practical significance of this second theorem becomes more evident in the case when U_1, \dots, U_p do not depend explicitly on time. This is stated as a corollary:

"If U_1, \dots, U_p do not depend explicitly on time, then, in order that $\chi(U_1, \dots, U_p)$ be definite, it is necessary and sufficient that at least one of the functions $U_i(x)$ assumes only positive values at all points for which $U_1(x) = \dots = U_{i-1}(x) = U_{i+1}(x) = \dots = U_p(x) = 0$, except at $x = 0$. Moreover if the last condition is satisfied by at least one of the functions $U_i(x)$, then it is satisfied by any other function".

This last result essentially simplifies the problem because from any (p-1) equations

$$U_1 = \dots = U_{i-1} = U_{i+1} = \dots = U_p = 0,$$

it is possible to express any $p-1$ variables, say x_{n-p+2}, \dots, x_n , in terms of x_1, \dots, x_{n-p+1} :

$$x_{n-p+2} = f_1(x_1, \dots, x_{n-p+1}), \dots, x_n = f_{p-1}(x_1, \dots, x_{n-p+1}).$$

If this can be done, then the problem of the definiteness of $\gamma(U_1, \dots, U_p)$ will be determined from the definiteness of the function

$$V(x_1, \dots, x_{n-p+1}) = U_1(x_1, \dots, x_{n-p+1}) f_1, \dots, f_{p-1}$$

with respect to the variables x_1, \dots, x_{n-p+1} .

Thus, the author in reference [25] has given certain conditions under which there exist functions of the first integrals which are definite.

The following theorem gives conditions under which no definite first integral can be constructed from U_1, U_2, \dots, U_p .

"If the known integrals do not depend explicitly on t and if they are of the form

$$U_i = (\underline{a}_i)_T \underline{x} + \text{higher order terms}$$

where the rank of $(\underline{a}_1, \dots, \underline{a}_p)$ is p , then no definite first integral can be constructed from U_1, \dots, U_p ".

The method of Chetaev [63] is as follows:

"If the given time-independent integrals are holomorphic functions of the variables, then the constants $\lambda_1, \dots, \lambda_p, C_1, \dots, C_p$ are selected in such a way that the expansion of the function

$$\gamma(U_1, \dots, U_p) = \lambda_1 U_1 + \dots + \lambda_p U_p + C_1 U_1^2 + \dots + C_p U_p^2$$

begins with a definite quadratic form".

By the first theorem of Pozharitskii such constants can be selected only when the function $\gamma(U_1, \dots, U_p)$ is definite. Thus, Pozharitskii's theory includes the work of Chetaev. From the above theory some guidelines concerning the choice of

the proper functions of first integrals are given.

It was the work of Chetaev in using a definite linear bundle of first integrals for Liapunov functions which helped the Russian mathematicians and engineers in their study of the stability analysis of complicated gyroscopes and other rigid body motions. There are many Russian papers concerning this application of Liapunov theory; an incomplete list of references is given in the back of this section, [16] to [56].

From reference [26] we now present an example of Chetaev's theory. This example concerns the stability of a heavy symmetrical gyroscope on gimbals; thus, there are three degrees of freedom, described by the Eulerian angles ϕ, γ, Θ . The principal moments of inertia for the rotor are $A_1 = B_1, C_1$; those for the inner gimbal ring are A_2, B_2, C_2 ; and the principal moment of inertia for the outer gimbal ring is C_3 . The defining equations for this system are Euler's equations of motion for rotating rigid bodies. The first three integrals of the equations of motion are

$$\begin{aligned}\dot{\phi} + u\dot{\gamma} &= r_0, \\ \dot{\gamma}(K_5 - K_6 u^2) + K_1 u &= K_2, \\ \dot{\Theta}^2 + \dot{\gamma}^2(K_5 - K_6 u^2) + K_3 u &= K_4,\end{aligned}$$

where $u = \cos \Theta$ and the K_i 's depend on the parameters of the system. The equilibrium position corresponds to the case when the rotor's axis is vertical; that is,

$$\begin{aligned}\Theta &= 0 \text{ or } u = 1, \dot{\Theta} = 0 \\ \dot{\phi} &= \dot{\phi}_0, \dot{\gamma} = \dot{\gamma}_0.\end{aligned}$$

The perturbation equations are obtained by introducing the following change of variables

$$\dot{\Theta} = x_1, \dot{\phi} = \dot{\phi}_0 + x_2, \dot{\gamma} = \dot{\gamma}_0 + x_3, u = 1 - x_4,$$

where the initial conditions of the perturbed motion are

$$\begin{aligned} r_{00} &= r_0 - R, \quad K_{10} = K_1 - K_1^*, \quad K_{20} = K_2 - K_2^*, \\ K_{40} &= K_4 - K_4^*. \end{aligned}$$

For the perturbation equations the above first integrals are

$$\begin{aligned} R &= x_2 + x_3 - \dot{\gamma}_0 x_4 - x_3 x_4, \\ K_2^* - K_1^* &= K_6 x_3 x_4^2 + 2K_6 x_3 x_4 + \\ &\quad - K_6 \dot{\gamma}_0 x_4^2 + (K_5 - K_6) x_3 + (2K_6 \dot{\gamma}_0 - K_1) x_4 \\ K_4^* &= x_1^2 - K_6 x_3^2 x_4^2 + 2K_6 x_3^2 x_4 - 2\dot{\gamma}_0 K_6 x_3 x_4^2 + \\ &\quad + (K_5 - K_6) x_3^2 + 4K_6 \dot{\gamma}_0 x_3 x_4 - K_6 \dot{\gamma}_0^2 x_4^2 + \\ &\quad + 2\dot{\gamma}_0 (K_5 - K_6) x_3 + (2\dot{\gamma}_0^2 K_6 - K_3) x_4. \end{aligned}$$

We now attempt to apply Chetaev's method of obtaining Liapunov functions; consider

$$\begin{aligned} V &= K_4^* + \alpha_1 (K_2^* - K_1^*) + \alpha_2 K_{20}^* + \\ &\quad + \alpha_3 R^2, \end{aligned}$$

where the α_i are arbitrary and $K_6^* x_4^2 + 1 - u^2 - 2x_4 = 0$. The \dot{V} is zero because V is a combination of first integrals. The linear terms in V can be made to vanish if $\alpha_1 = -2\dot{\gamma}_0$ and $\alpha_2 = K_1 \dot{\gamma}_0 - \frac{1}{2} K_3$.

Then the Liapunov function will consist of a quadratic form and third - and fourth - order terms. The quadratic form is positive definite and thus the stability of the gyroscope with respect to $\theta, \dot{\theta}, \phi, \dot{\phi}, \dot{\gamma}$ is guaranteed if the following inequalities are satisfied:

$$\begin{aligned} \alpha_3 &> 0, \quad K_5 - K_6 > 0, \quad \dot{\gamma}_0 K_1 - 1/2 K_3 > 0, \\ \dot{\gamma}_0 K_1 - 1/2 K_3 - \dot{\gamma}_0^2 K_6 &> 0. \end{aligned}$$

RUMIANTSEV'S WORK

In reference [20], Ruminatsev considers an arbitrary holonomic mechanical system, where q_1, \dots, q_n are the independent generalized coordinates and $\dot{q}_1, \dots, \dot{q}_n$ are the generalized velocities. The unperturbed solution of the system is given as

$$q_i = f_i(t), i = 1, \dots, n, \quad (1)$$

where the initial values are $q_{i0} = f_i(t_0)$ and $\dot{q}_{i0} = \dot{f}_i(t_0)$. For the perturbed motion, let

$$q_{i0} = f_i(t_0) + \epsilon_i, \dot{q}_{i0} = \dot{f}_i(t_0) + \dot{\epsilon}_i$$

where ϵ_i and $\dot{\epsilon}_i$ are real constants designated as perturbations. Since the force system remains unchanged, these constants define completely the perturbed motion.

The values, q_i and \dot{q}_i , are now replaced for the perturbed motion by

$$q_i = f_i(t) + x_i, \dot{q}_i = \dot{f}_i(t) + x_{n+i}$$

where $x_i(t)$, $i = 1, \dots, 2n$, are the variations of the variables q_i and \dot{q}_i .

The defining equations of the perturbed motion can be written as

$$\dot{\underline{x}} = \underline{F}(t, \underline{x}), \quad (2)$$

where \underline{x} and \underline{F} are $2n$ - vectors.

We assume that \underline{F} is such that a unique solution exists for every $t \geq t_0$ and that $\underline{F}(t, 0) = 0$.

We are interested in the stability of the unperturbed motion in (1) with respect to certain real continuous functions Q_1, \dots, Q_k of the variables x_i and time t . For the unperturbed motion the Q_i 's are known functions of time, $g_i(t)$. For the perturbed motion the Q_i 's are functions of t and the perturbations ϵ_i and $\dot{\epsilon}_i$. Considering the differences $y_s = Q_s - g_s$, Liapunov called the unperturbed motion (1) stable with respect to the quantities

Q_1, \dots, Q_k , if for all L_s , there exist $E_i > 0$ and $\dot{E}_i > 0$ such that for any $\epsilon_i, \dot{\epsilon}_i$ satisfying the conditions

$$|\epsilon_i| \leq E_i, \quad |\dot{\epsilon}_i| \leq \dot{E}_i$$

for any $t \geq t_0$, the following inequalities hold: $|y_s| < L_s$; $s = 1, \dots, k$. Further, we assume that for any set of real values of $\epsilon_i, \dot{\epsilon}_i$, numerically sufficiently small, there corresponds a certain set of real initial values y_{s0} of the variables y_s such that for a sufficiently small $A > 0$,

$$y_{10}^2 + \dots + y_{k0}^2 < A,$$

if $|\epsilon_i| \leq E_i$ and $|\dot{\epsilon}_i| \leq \dot{E}_i$. The converse of this last statement also is assumed to hold.

Since the y_s are related to t and the x_j , then the region of variation of the real variables t, x_1, \dots, x_{2n}

$$t \geq t_0, \quad x_1^2 + \dots + x_{2n}^2 \leq H, \quad (3)$$

where t_0 and $H > 0$ are constants, will correspond to the region

$$t \geq t_0, \quad y_1^2 + \dots + y_k^2 \leq H_1, \quad (4)$$

of variation of the variables t, y_s , where $H_1 > 0$ is a constant.

We shall assume (2) has a first integral

$$\phi(\underline{x}, t) = \text{constant}, \quad (5)$$

which is a real, continuous, bounded function of its variables in the region defined by (3).

THEOREM

"If the differential equations of the perturbed motion (2) admit a first integral (5) and it is possible to find a positive definite function

$\Phi(y_1, \dots, y_k, t)$ such that the inequality

$$\bar{\phi}(y_1, \dots, y_k, t) \leq \phi(x_1, \dots, x_{2n}, t)$$

is satisfied for all values of t , y_s in region (4), then the unperturbed motion (1) is stable with respect to the quantities Q_1, \dots, Q_k ."

As an example of this theorem, Rumiantsev considers the well known problem of stability of rotation about a vertical axis of a heavy rigid body in the case of Lagrange [20]. We will not repeat his example in this report since the above discussion was introduced because of its important application in "continuous media" problems. A short discussion of this fact follows.

The theory given above is useful in the application of the second method of Liapunov to the problems of stability of motion of continuous media with respect to a finite number of parameters, which describe the motion through certain integral relationships. Examples of such parameters could be the coordinates of the center of gravity of a bounded volume of a continuous medium, or projections of its linear momentum on certain axes, or similar quantities, whose variations with time are described by ordinary differential equations. The stability of motion of a continuous medium with respect to the above mentioned parameters will be called the conditional stability of motion of a continuous medium. An example of this theory is given in the compendium, example, No. 23.

In passing, we make note of a "somewhat analogous" paper written by Pozharitskii, [40]. This paper deals with the asymptotic stability of dynamic systems with partial dissipation. That is, in a mechanical system it may be sufficient to introduce damping in only part of the coordinates describing the system in order to obtain asymptotic stability. First integrals are used in his discussion of this concept.

INFANTE'S WORK

In references [57] and [58], Infante considers the second order, autonomous system with an equilibrium point at the origin. The method of generating Liapunov functions developed in these papers is an outgrowth of the original geometric considerations in Liapunov's theory.

The technique proposed by Infante is based on the accessibility and availability of a nontrivial time-independent integral of the second-order system which can be used in stability studies. If an integral, with the proper stability properties, cannot be found, a modified system is constructed which satisfies two criteria; one, the system has a first integral with admissible stability properties and two, the important qualitative properties in the original system are retained in the modified system. That is, we seek a "nearby" system. The integral of the nearby system is considered to be a candidate for a Liapunov function of the original system. We will now consider the development of Infante's method; and in the compendium at the end of this section, we include many examples of this method of constructing Liapunov functions.

Consider the second order system described by

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2),\end{aligned}\tag{1}$$

where any time-independent integral

$$h(x_1, x_2) = C,$$

which might exist, must satisfy the following:

$$\frac{\partial h}{\partial x_1} \dot{x}_1 + \frac{\partial h}{\partial x_2} \dot{x}_2 = \frac{\partial h}{\partial x_1} f_1 + \frac{\partial h}{\partial x_2} f_2 = 0.\tag{2}$$

A sufficient condition for the existence of $h = C$ is

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0. \quad (3)$$

Since most systems do not satisfy equation (3), the system in (1) must be modified.

For simplicity, let us replace (1) by the following more common state variable form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_2(x_1, x_2). \end{aligned} \quad (4)$$

If $\frac{\partial f_2}{\partial x_2} = 0$, then (4) has an integral which could be used in stability work.

But if $\frac{\partial f_2}{\partial x_2} = f_3(x_1, x_2) \neq 0$, then no integral can be found and a "nearby system" is defined in such a way that this new system has a readily accessible first integral. The first nearby system we try is

$$\begin{aligned} \dot{x}_1 &= x_2 - \int_0^{x_1} f_3(x_1, x_2) dx_1 \\ \dot{x}_2 &= f_2(x_1, x_2). \end{aligned} \quad (5)$$

By (3), system (5) certainly possesses a first integral but there is no assurance that the qualitative properties of (4) are retained. The system (5) is modified by adding arbitrary functions which must satisfy certain geometrical criteria in order to retain the qualitative properties of (4). The new "nearby" system is

$$\begin{aligned} \dot{x}_1 &= x_2 - \int_0^{x_1} f_3(x_1, x_2) dx_2 + f_4(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) + f_5(x_1, x_2), \end{aligned} \quad (6)$$

where $\frac{\partial f_4}{\partial x_1} = - \frac{\partial f_5}{\partial x_2}$. System (6) satisfies equation (3) and the arbitrary functions f_4 and f_5 are chosen such that the major qualitative properties of (4) are retained. The geometrical relationship between systems (4) and (6) is specified by the third component, x^* , of the cross product of the "flows" in phase space; that is (x_1, x_2) of system (4) "crossed with" (x_1, x_2) of system (6).

The candidate for the Liapunov function of system (4) is chosen to be the first integral of system (6). The "nearness" of system (6) to that of system (4) is determined by the algebraic sign of x^* , if both the vectors (x_1, x_2) , for systems (4) and (6), rotate clockwise in the phase plane. If $x^* \geq 0$, then the integral of (6) is a Liapunov function of (4) and will also give an estimate of the region of asymptotic stability about $\underline{x} = \underline{0}$.

The claims made by Infante in support of his method of generating Liapunov functions are as follows:

- (1) Simplicity of application and requiring no deep insight into the problem;
 - (2) the estimates of the domain of asymptotic stability are very good for wide ranges of the parameters in the differential equations;
 - (3) and the flexibility of the method, due to the choice of f_4 and f_5 .
- A disadvantage of the method is that a poor choice of f_4 and f_5 may contribute to very conservative estimates of the region of asymptotic stability.

WALKER'S WORK

The work of Walker, [60] and [61], is an extension of Infante's work to n^{th} order, nonlinear, autonomous systems. He considers systems of the form

$$\frac{d^n x}{dt^n} + g(x, \dot{x}, \dots, \frac{d^{n-1} x}{dt^{n-1}}) = 0, \quad (1)$$

which possess (n-1) first integrals

$$U_i(x, \dot{x}, \dots, \frac{d^{n-1}x}{dt^{n-1}}, t) = K_i, \quad i=1, 2, \dots, n-1. \quad (2)$$

Equation (1) can also be written as a system of n - first order equations in state variable notation:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \dots, x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n). \end{aligned} \quad (3)$$

These equations are equivalent to

$$dt = \frac{dx_1}{f_1} = \frac{dx_2}{f_2} = \dots = \frac{dx_n}{f_n}. \quad (4)$$

Eliminating the explicit dependence of the equation in (4) on time t, we get n-1 first order differential equations in the n variables. The solution of these equations are (n-1) first integrals of the system in (1); these are denoted by:

$$\begin{aligned} \psi_1(x) &= c_1 \\ &\dots \dots \dots \\ \psi_{n-1}(x) &= c_{n-1}, \end{aligned} \quad (5)$$

where x_T is (x_1, \dots, x_n) . The integrals in (5) may not all be independent; but each integral in (5) must satisfy.

$$d\psi_i = \sum_{j=1}^n \frac{\partial \psi_i}{\partial x_j} dx_j = \sum_{j=1}^n \frac{\partial \psi_i}{\partial x_j} f_j = 0. \quad (6)$$

Thus, if we solve the equation (6) for all the (n-1) first integrals our stability problems "are over". The trouble here is that equation (6), in general, is very difficult to solve for ψ_i .

One case in which it is possible to obtain a first integral is when the f_1 are of the following form:

$$\begin{aligned}\dot{x}_1 &= \theta_1 \frac{\partial H}{\partial x_2}, \\ \dot{x}_2 &= \theta_2 \frac{\partial H}{\partial x_3} - \theta_1 \frac{\partial H}{\partial x_1}, \\ &\vdots \\ \dot{x}_{n-1} &= \frac{\partial H}{\partial x_n} - \theta_{n-2} \frac{\partial H}{\partial x_{n-2}}, \\ \dot{x}_n &= - \frac{\partial H}{\partial x_{n-1}},\end{aligned}\tag{7}$$

where $H, \theta_1, \dots, \theta_{n-2}$ are certain functions of the state variables.

Combining (6) and (7) gives

$$\begin{aligned}\frac{\partial \psi_1}{\partial x_1} (\theta_1 \frac{\partial H}{\partial x_2}) + \frac{\partial \psi_1}{\partial x_2} (\theta_2 \frac{\partial H}{\partial x_3} - \theta_1 \frac{\partial H}{\partial x_1}) + \dots \\ \dots + \frac{\partial \psi_1}{\partial x_{n-1}} \left(\frac{\partial H}{\partial x_n} - \theta_{n-2} \frac{\partial H}{\partial x_{n-2}} \right) + \frac{\partial \psi_1}{\partial x_n} \left(- \frac{\partial H}{\partial x_{n-1}} \right) = 0.\end{aligned}\tag{8}$$

If $\psi_1 = H$ in (8), then we see that the equation is identically satisfied and at least one integral of (7) is H itself. But the disadvantage here is that very few n^{th} order systems have the form given in (7). One example is

$$\ddot{x} + K \dot{x} = 0,$$

or

$$\begin{aligned}\dot{x}_1 &= x_2 = \theta_1 \frac{\partial H}{\partial x_2}, \\ \dot{x}_2 &= x_3 = \frac{\partial H}{\partial x_3} - \theta_1 \frac{\partial H}{\partial x_1}, \\ \dot{x}_3 &= -Kx_2 = - \frac{\partial H}{\partial x_2},\end{aligned}$$

In this example $\Theta_1 = 1/K$ and the $(n-1)$ independent integrals are

$$\gamma_1 = 1/2 (K x_2^2 + x_3^2) = c_1,$$

$$\gamma_2 = K/2 (x_2^2 - 2x_1x_3 - Kx_1^2) = c_2.$$

From the above theory concerning first integrals of the n^{th} order system, (1), and from the work of Infante, Walker derived the following method for constructing Liapunov functions by obtaining first integrals of "nearby systems".

The state variable form of equation (1) is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= -g(x_1, x_2, \dots, x_n). \end{aligned} \tag{9}$$

A modified system is now chosen which has the same form as (7):

$$\begin{aligned} \Theta_1 \frac{\partial H}{\partial x_2} &= \dot{x}_1 = x_2 + g_1(x) \\ \Theta_2 \frac{\partial H}{\partial x_3} - \Theta_1 \frac{\partial H}{\partial x_1} &= \dot{x}_2 = x_3 + g_2(x) \\ &\vdots \\ \frac{\partial H}{\partial x_n} - \Theta_{n-2} \frac{\partial H}{\partial x_{n-2}} &= \dot{x}_{n-1} = x_n + g_{n-1}(x) \\ \frac{\partial H}{\partial x_{n-1}} &= \dot{x}_n = -g(x), \end{aligned} \tag{10}$$

where the g_i are restricted by the left sides of the equations. Note that the last equation in both systems (9) and (10) are the same; this is done so as to retain as much similarity between the systems as possible. A more

condensed form for (10) is obtained if the arbitrary g_i are written such that

$$\begin{aligned} \frac{\partial H}{\partial x_1} &= h_1(\underline{x}), \\ \frac{\partial H}{\partial x_2} &= h_2(\underline{x}), \\ &\text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \frac{\partial H}{\partial x_{n-2}} &= h_{n-2}(\underline{x}), \end{aligned} \tag{11}$$

$$\frac{\partial H}{\partial x_{n-1}} = g(\underline{x}),$$

$$\frac{\partial H}{\partial x_n} = x_n + h_n(\underline{x}),$$

where the h_i are conveniently defined as

$$h_i = \int \frac{\partial}{\partial x_i} (g(\underline{x})) dx_{n-1}. \tag{12}$$

Specification of the h_i in this manner does not completely determine the "nearby" system but the finding of a specific system is not our objective. Our objective is to find a usable integral H which is also a Liapunov function of (9). An integral H of the system in (11) and (12) may not be a Liapunov function of (9) and thus we modify the equations in (11) in the following manner:

$$\begin{aligned} \frac{\partial V}{\partial x_1} &= \frac{\partial H}{\partial x_1} + f_1, \\ \frac{\partial V}{\partial x_2} &= \frac{\partial H}{\partial x_2} + f_2, \\ &\text{---} \text{---} \text{---} \text{---} \text{---} \end{aligned} \tag{13}$$

$$\frac{\partial V}{\partial x_1} = \frac{\partial H}{\partial x_1} + f_1,$$

$$\frac{\partial V}{\partial x_2} = \frac{\partial H}{\partial x_2} + f_2,$$

$$\frac{\partial V}{\partial x_n} = \frac{\partial H}{\partial x_n} + f_n,$$

(13)

where $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$. Note, that these f_i are different from the f_i in (3). In conclusion we say that if the f_i can be chosen such that V is a Liapunov function of (9), a new method of generating Liapunov functions has been developed. The $\frac{\partial H}{\partial x_i}$ in (13) are defined by (11) and (12). Many examples of this procedure are given in the compendium of examples.

The advantages of this method according to Walker are:

- (1) more than half of the gradient, ∇V , of the final Liapunov function is developed automatically,
- (2) hints to further modifications of the technique are given by the procedure itself,
- (3) good balance between automatic generation of functions and flexibility of application.

The usual disadvantages of other methods are also present here:

- (1) restricted to autonomous systems,
- (2) results are difficult to obtain for fourth and higher order systems.

COMPENDIUM OF EXAMPLES

This set of examples considers four types of integral methods used to construct Liapunov functions. These types are:

- (1) the use of a definite first integral for a Liapunov function,
- (2) the use of a combination of first integrals,
- (3) Infante's integral method for second order systems,
- (4) Walker's integral method for higher order systems.

These examples also point out the variety of different physical problems which can be analyzed by Liapunov's method; such as, electrical networks, mechanical vibrations, control systems, nuclear reactor dynamics, magneto-hydrodynamics, and others. The extent of the region of asymptotic stability is also approximated in many of the following examples.

Example 1, [1] Generalized LRC - Circuit

This example is a generalization of the LRC equation of electricity:

$$\ddot{x} + f(x) \dot{x} + g(x) = 0.$$

In LaSalle's discussion of the region of asymptotic stability, in reference [1], he simplified the nonlinearities in the following way:

- 1) f and g are polynomials,
- 2) f is even and g is odd,
- 3) g acts like a straight line through the origin, and
- 4) g is monotone increasing with x .

We introduce the integrals

$$F(x) = \int_0^x f(x) \, dx,$$
$$G(x) = \int_0^x g(x) \, dx,$$

where F is odd and G is even, and $F(0) = G(0) = 0$.

Consider the equivalent system

$$\dot{x} = y - F(x),$$

$$\dot{y} = -g(x).$$

Since F and g are polynomials, existence of solutions is guaranteed. The equilibrium solution is the origin $(0,0)$.

As a candidate for a Liapunov function, we choose the total energy of the system when there is no dissipation, $f \equiv 0$; that is,

$$V = 1/2 y^2 + G(x).$$

The time derivative of the V - function is

$$\dot{V} = y\dot{y} + g(x)\dot{x} = -yg(x) + g(x)y - g(x)F(x) = -g(x)F(x).$$

If there exist positive constants a and L such that

$$g(x)F(x) > 0 \quad \text{for} \quad |x| < a, \quad x \neq 0$$

and

$$G(x) < L \quad \text{for} \quad |x| < a,$$

then the bounded region \mathcal{U}_L , defined by $V(x,y) < L$, is a measure of the extent of asymptotic stability of the system about the origin.

Example 2, [1] Van der Pol's Equation

A special case of the previous example is the Van der Pol equation:

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0, \quad \epsilon > 0,$$

or equivalently,

$$\dot{x} = y - \epsilon \left(\frac{x^3}{3} - x \right),$$

$$\dot{y} = -x.$$

The only equilibrium point is the origin and the linear approximation shows that it is unstable. If t is replaced by $-t$, then the phase plane

trajectories remain the same but the orientation is reversed. The origin is then asymptotically stable. This same effect is obtained if we let $\epsilon < 0$ and retain the original orientation of t .

Thus, as in the previous example, we have for V :

$$V = y^2/2 + G(x) = \frac{x^2 + y^2}{2}$$

where $f(x) = \epsilon(x^2 + 1)$ and $g(x) = x$.

The time derivative of V is

$$\dot{V} = -\epsilon x^2 (x/3 - 1), \quad \epsilon < 0.$$

Thus, $\dot{V} \leq 0$ for $x^2 \leq 3 = a^2$. Taking $L = 3/2$, we find that the region of asymptotic stability chosen in this manner is defined by $x^2 + y^2 < 3$.

Example 3, [1] A Second Order Example

Consider the second order equation given by

$$\ddot{x} + a\dot{x} + 2bx + 3x^2 = 0; \quad a, b > 0,$$

or its equivalent

$$\dot{x} = y$$

$$\dot{y} = -2bx - ay - 3x^2.$$

The equilibrium solutions are $(0,0)$ and $(-2b/3, 0)$. By linear approximation, the point $(0,0)$ is asymptotically stable and the point $(-2/3 b, 0)$ is an unstable "saddle point". By Liapunov theory we can construct a region of asymptotic stability about the origin. The total energy of the corresponding undamped ($a = 0$) system is chosen as a Liapunov function:

$$V = y^2/2 + bx^2 + x^3,$$

where

$$\dot{V} = -a y^2.$$

Thus, a region of asymptotic stability about (0,0) is defined by the inequality

$$V = y^2 / 2 + bx^2 + x^3 < L = \left[\frac{4}{27} b^3 \right],$$

where $V = \frac{4b^3}{27}$ forms a closed loop containing (0,0) and passing through $(-2/3b, 0)$.

Example 4, [1,2] Lienard's Equation

For Lienard's equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

we assume that

$$(1) \quad xg(x) > 0, \quad x \neq 0, \quad g(0) = 0,$$

$$(2) \quad f(x) > 0, \quad x \neq 0, \quad f(0) = 0$$

$$(3) \quad G(x) = \int_0^x g(x) dx \rightarrow \infty \text{ as } \|x\| \rightarrow \infty.$$

Thus, we assume that the potential energy $G(x)$ is positive definite and that at $x = 0$ is its minimum; the potential energy approaches infinity with $|x|$; and the damping is always positive.

An equivalent system is

$$\dot{x} = y,$$

$$\dot{y} = -g(x) - yf(x).$$

The Liapunov function is taken to be the total energy of the system

(where $f = 0$);

$$V = 1/2 y^2 + G(x),$$

where

$$\dot{V} = -f(x) y^2 \leq 0.$$

Since

$$V \rightarrow \infty \text{ as } x^2 + y^2 \rightarrow \infty, \text{ then all}$$

solutions are bounded for $t \geq 0$. $\dot{V} \equiv 0$ only for the equilibrium solution of the system. Thus, the whole region of space is the domain of asymptotic stability; that is, the system is completely stable.

We also note that if $f(x) \equiv 0$ for all x then we can conclude from above that the system is stable; and if $f(x) < 0$ for $x \neq 0$, then the system is unstable at the origin.

Example 5, [2] Lewis Servomechanism

The defining equation of the Lewis servomechanism is a special case of Lienard's equation:

$$\ddot{x} + 2 \int_0^x (1 - a |x|) \dot{x} + x = 0,$$

where \int is a system parameter. In this equation

$$f(x) = 2 \int_0^x (1 - a |x|),$$

$$g(x) = x.$$

The state variable formulation is obtained by choosing y as

$$y = \dot{x} + \int_0^x f(x) dx = \dot{x} + F(x),$$

where the time derivative of y gives

$$\dot{y} = \ddot{x} + f(x)\dot{x} = -g(x) = -x.$$

Thus, the system can be described as

$$\dot{x} = y - F(x),$$

$$\dot{y} = -g(x) = -x.$$

The same Liapunov function is chosen as in the previous example:

$$V = y^2/2 + \int_0^x g(x) dx,$$

where

$$\begin{aligned}\dot{V} &= y \dot{y} + \dot{x} g(x) \\ &= -g(x)F(x) \\ &= -x \int_0^x 2 \zeta (1 - a|x|) dx.\end{aligned}$$

In the vicinity of the origin, the rest position, \dot{V} is negative semidefinite if $\zeta > 0$. Therefore the system is asymptotically stable.

As a numerical example, say that $2\zeta = a = 1$. Then

$$F(x) = x - \frac{x|x|}{2}, \quad G(x) = \frac{x^2}{2}, \text{ and } V = 1/2 (x^2 + y^2)$$

where $\dot{V} = -x^2/2 (2 - |x|)$. Thus, \dot{V} is negative semidefinite if

$|x| \leq 2$. Therefore for any solution starting inside, the circle $y^2 + x^2 = 4$ in the xy-plane, it approaches $(0,0)$ as $t \longrightarrow \infty$. Any limit cycle of the system lies outside this circle.

Example 6, [2] Rotating Rigid Body

The Euler equations for angular motions of a rigid body in space are

$$A\dot{p} = (B-C) qr,$$

$$B\dot{q} = (C-A) pr,$$

$$C\dot{r} = (A-B) pq,$$

where $A < C < B$. The A, B, C are moments of inertia and the p, q, r are angular velocities about the x, y, z - axes, respectively. This system could represent an artificial satellite. The motion which we analyze is the following; assume that the satellite is rapidly spinning around the z - axis, that is,

$$|r| \gg 1,$$

$$|p| \ll 1,$$

$$|q| \ll 1.$$

The spinning of the body exhibits a certain gyroscopic rigidity but the motion is unstable. The proof of this statement is as follows.

Let the angular velocity about the z - axis be expressed as the sum of a steady spin rate R_0 plus a small perturbation r_* , $r = r_* + R_0$. The equations of motion become

$$A\dot{p} = (B-C) q (R_0 + r_*) ,$$

$$B\dot{q} = (C-A) p (R_0 + r_*) ,$$

$$C\dot{r}_* = (A-B) pq .$$

If p and q are assumed to be positive or zero together in the region of interest, then we can consider the positive definite form

$$V = pq + r_*^2 ,$$

where

$$\begin{aligned} \dot{V} &= \dot{p} q + p \dot{q} + 2 r_* \dot{r}_* \\ &= \left[\frac{C-A}{B} p^2 + \frac{B-C}{A} q^2 \right] R_0 + \\ &\quad + \left[\frac{C-A}{B} p^2 + \frac{B-C}{A} q^2 - 2 \frac{B-A}{C} pq \right] r_* . \end{aligned}$$

The equilibrium solution of this new system is $(p, q, r_*) = (0, 0, 0)$.

Thus, when we are sufficiently close to the origin, the second "square

bracket" term in \dot{V} is dominated by the first "bracketed" term. Therefore if

$R_0 > 0$, V and \dot{V} have the same sign and the system is unstable. If $R_0 < 0$ and

p and q have opposite signs, then the system is also unstable. The con-

clusion is that a rapid spin about the axis of intermediate moment of inertia can not be maintained.

Example 7, [3] Passive Nonlinear Network

The following example is concerned with a passive nonlinear network whose elements are nonlinear inductors, capacitors, and resistors. In this circuit there are no internal sources, mutual inductances, or ideal transformers. The energy storage elements are the inductors, $j = 1, \dots, m$, and the capacitors, $j = m + 1, \dots, n$; the remaining elements in the circuit are resistors. The notation is as follows:

$$\begin{aligned} e_j &= \text{voltage of the } j^{\text{th}} \text{ element} \\ i_j &= \text{current in the } j^{\text{th}} \text{ element} \\ \psi_j &= \text{flux in the } j^{\text{th}} \text{ element} \\ q_j &= \text{charge in the } j^{\text{th}} \text{ element.} \end{aligned}$$

The governing equations of the circuit are:

(1) for the inductors; $j = 1, \dots, m$

$$\begin{aligned} i_j &= f_j(\psi_j) = \text{nonlinear function of } \psi_j \\ e_j &= \dot{\psi}_j, \end{aligned}$$

(2) for the capacitors; $j = m + 1, \dots, n$

$$\begin{aligned} e_j &= f_j(q_j) = \text{nonlinear function of } q_j \\ i_j &= \dot{q}_j. \end{aligned}$$

Thus, the state variables, x_1, \dots, x_n , are $\psi_1, \dots, \psi_m, q_{m+1}, \dots, q_n$.

The equilibrium point is assumed to be $\underline{x} = \underline{0}$, where $f_j(0) = 0$, ($j = 1, \dots, n$).

As a candidate for a Liapunov function for this passive resistive network, we choose the stored energy of the system:

$$\begin{aligned}
 E = V(\underline{x}) &= \sum_{j=1}^n \int_0^t e_j i_j dt \\
 &= \sum_{j=1}^m \int_0^{\gamma_j} f_j(\gamma_j) d\gamma_j + \sum_{j=m+1}^n \int_0^{q_j} f_j(q_j) dq_j \\
 &= \sum_{j=1}^n \int_0^{x_j} f_j(x_j) dx_j,
 \end{aligned}$$

where

$$\dot{V}(\underline{x}) = \sum_{j=1}^n e_j i_j < 0,$$

for a passive resistive network. Thus, $V(\underline{x})$ is a Liapunov function if

$$(1) \quad x_j f_j(x_j) > 0,$$

$$(2) \quad \int_0^{\infty} f_j(x_j) dx_j = \infty.$$

Therefore, the system is asymptotically stable in the large.

Example 8, [1] Rigid Body Motion

This example will illustrate the use of linear approximation and the use of first integrals. Consider the following system which is frequently seen in the study of the motion of rigid bodies:

$$\dot{x}_1 = Ax_2 (x_3 - a),$$

$$\dot{x}_2 = Bx_1 (x_3 - b),$$

$$\dot{x}_3 = x_1 x_2,$$

where A, B, a, b are constants and x_1, x_2, x_3 are the coordinates of the rigid body.

where A, B, a and b are constants. There are three equilibrium states

$$E_1 : x_1 = L, x_2 = 0, x_3 = b$$

$$E_2 : x_1 = 0, x_2 = m, x_3 = a$$

$$E_3 : x_1 = 0, x_2 = 0, x_3 = m$$

where L, m, n are arbitrary constants.

Stability of E_1

We change the coordinates so that E_1 is at the origin. The equations of transformation are

$$y_1 = x_1 - L, y_2 = x_2, y_3 = x_3 - b.$$

The new system equations are

$$\dot{y}_1 = A(b-a)y_2 + Ay_2 y_3,$$

$$\dot{y}_2 = B(y_1 + L)y_3,$$

$$\dot{y}_3 = (y_1 + L)y_2.$$

The characteristic equation for the linear approximation is

$$\begin{vmatrix} -\lambda & A(b-a) & 0 \\ 0 & -\lambda & LB \\ 0 & L & -\lambda \end{vmatrix} = \lambda (L^2 B - \lambda^2) = 0.$$

The characteristic roots are 0, $L\sqrt{B}$, $-L\sqrt{B}$. For physical reasons assume $L \neq 0$; if $L = 0$, then E_1 and E_3 coincide.

If $B > 0$, then one of the roots is positive, and E_1 is unstable. If

$B < 0$, then we have the critical case and we must look at the nonlinear terms.

There are two obvious first integrals; namely,

$$V_1 = y_2^2 - By_3^2 = C_1$$

and

$$V_2 = (y_1 + L)^2 - A (y_3 + b - a)^2 = C_2,$$

where $\dot{V}_1 = \dot{V}_2 = 0$. If $B < 0$, then V_1 is positive definite in y_2 and y_3 and thus we have stability in these two variables. This means that if y_2 and y_3 are initially small, then they remain small. From the second integral we see that if y_3 remains small then y_1 must also remain small.

Thus,

(1) E_1 is stable if $B < 0$,

(2) E_2 is unstable if $B > 0$.

Stability of E_2

By symmetry we see that

(1) E_2 is stable if $A < 0$,

(2) E_2 is unstable if $A > 0$.

Stability of E_3

Change the coordinates by using the following

$$y_1 = x_1, y_2 = x_2, y_3 = x_3 - m.$$

Thus, E_3 is at the origin in the new system:

$$\dot{y}_1 = Ay_2 (y_3 - a + m),$$

$$\dot{y}_2 = By_1 (y_3 - b + m),$$

$$\dot{y}_3 = y_1 y_2.$$

The characteristic roots corresponding to the linear approximation are

$$0, \pm \sqrt{AB (m-a)(m-b)}.$$

If $\{AB (m-a)(m-b)\} > 0$, then E_3 is unstable. If $\{AB (m-a)(m-b)\} < 0$,

then we have a critical case and the nonlinear terms must be considered.

An integral of the system is

$$V = -B(m-b)y_1^2 + A(m-a)y_2^2 + AB(a-b)y_3^2 = C,$$

where $\dot{V} = 0$. Thus, we conclude

(1) E_3 is unstable if $\{AB(m-a)(m-b)\} > 0$,

(2) E is stable if $\{AB(m-a)(m-b)\} < 0$ and $\{A(a-b)(m-b)\} < 0$,

(3) this analysis fails if $\{AB(m-a)(m-b)\} < 0$
and $\{A(a-b)(m-b)\} > 0$.

Example 9, [4] Rigid Body Motion

This example is concerned with the use of a Liapunov function in determining the stabilization scheme which will achieve zero spin for a satellite in finite time. The time to reach zero rotation is proportional to the square root of the initial rotational energy.

The equations of motion of the satellite are

$$I_1 \dot{\omega}_1 = (I_2 - I_3) \omega_2 \omega_3 + T_1,$$

$$I_2 \dot{\omega}_2 = (I_3 - I_1) \omega_1 \omega_3 + T_2,$$

$$I_3 \dot{\omega}_3 = (I_1 - I_2) \omega_1 \omega_2 + T_3,$$

where ω_i is the rate about the i^{th} principal inertia axis, I_i is the moment of inertia about the i^{th} axis, and T_i is the torque input about the i^{th} axis.

Let the candidate for the Liapunov function be the total rotational energy:

$$V = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2).$$

The time derivative of V is

$$\dot{V} = T_1 \omega_1 + T_2 \omega_2 + T_3 \omega_3.$$

If we choose the input torques to be

$$T_i = \frac{-\alpha}{2} \frac{I_i \omega_i}{\sqrt{V}}$$

then we find

$$\dot{V} = -\alpha \sqrt{V}$$

and

$$t - t_0 = -\frac{2}{\alpha} \left[\sqrt{V(x(t))} - \sqrt{V(x(t_0))} \right].$$

Thus the time to reach zero rotation is

$$T - t_0 = \frac{2}{\alpha} \sqrt{V(x_0)},$$

since $V(0) = 0$ at zero rotation.

Example 10, [5,6] Lagrange's Theorem

Lagrange's theorem on the stability of the equilibrium point of an n - degree of freedom system can be proved through the use of Liapunov functions. Let q_i be the generalized positional coordinates and P_i be the generalized moments of this holonomic conservative system. The potential energy, $V = V(q)$, is a positive definite function of q . The kinetic energy, $T = T(q, p)$, is a positive definite quadratic form in p with coefficients analytic in q . The equations of motion of this system are:

$$\frac{d}{dt} \left\{ \frac{\partial T}{\partial p_i} \right\} - \frac{\partial T}{\partial q_i} = -\frac{\partial V}{\partial q_i}, \quad (i = 1, \dots, n)$$

$$\frac{dq_i}{dt} = p_i, \quad (i = 1, \dots, n).$$

Define the Hamiltonian to be $H = V + T$. This will be our candidate for a Liapunov function. By definition of T and V , we have that H is positive definite with respect to the equilibrium point $(p, q) = (0, 0)$. The time derivative of H along the trajectories of the system is :

$$\begin{aligned}\dot{H} &= \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i, \text{ (sum on } i) \\ &= \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} = 0,\end{aligned}$$

from the canonical form of the equations of motion. Thus, the equilibrium position is stable.

Example 11, [5] Instability Theorem

Dealing with the same conservative system, we now prove an instability theory. Let T be the same as in the previous example, but V now is negative definite at the rest or equilibrium point. The canonical equations of motion are as before:

$$\begin{aligned}\dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i},\end{aligned}$$

where $H = T + V$. We now expand T and $-V$ in the following forms:

$$-V = U_m(q) + U_{m+1}(q) + \dots,$$

where $U_i(q)$ are i^{th} degree homogeneous forms in q , and $m \geq 2$, and

$$2T = \sum_{\alpha, \beta=1}^n \left\{ a_{\alpha\beta} p_{\alpha} p_{\beta} \right\} + \sum_{\alpha, \beta=1}^n A_{\alpha\beta}(q) p_{\alpha} p_{\beta},$$

where $a_{\alpha\beta}$ are constants, and $A(0) = 0$. Since $T > 0$ if $p \neq 0$ and for any q , then $\sum a_{\alpha\beta} p_{\alpha} p_{\beta}$ is positive definite.

The candidate for a "Liapunov function" is

$$V_1 = \sum_1^n p_{\alpha} p_{\beta},$$

where V_1 has a variable sign. The time derivative of V_1 is

$$\begin{aligned}
 \dot{V}_1 &= \sum_{i=1}^n (p_i \dot{q}_i + \dot{p}_i q_i) \\
 &= \sum_{i=1}^n \left(p_i \frac{\partial H}{\partial p_i} - q_i \frac{\partial H}{\partial q_i} \right) \\
 &= \sum_{i=1}^n \left(p_i \frac{\partial T}{\partial p_i} + q_i \frac{\partial U_m}{\partial q_i} + q_i \frac{\partial U_{m+1}}{\partial q_i} + \dots \right) + \\
 &\quad - \sum_{i,j,K=1}^n q_j \frac{\partial A_{ik}}{\partial q_j} p_i p_k .
 \end{aligned}$$

Apply Euler's theorem on homogeneous functions to give the following:

$$\begin{aligned}
 V_1 &= \left\{ \sum_{i,j=1}^n a_{ij} p_i p_j + \sum_{i,j=1}^n \left(A_{ij} - \sum_{k=1}^n q_k \frac{\partial A_{ij}}{\partial q_k} \right) p_i p_j \right\} + \\
 &\quad + \left\{ m U_m + (m+1) U_{m+1} + \dots \right\} .
 \end{aligned}$$

The first bracketed term is positive definite with respect to \underline{p} because the variable coefficient $\left(A_{ij} - \sum_{k=1}^n q_k \frac{\partial A_{ij}}{\partial q_k} \right)$, can be made sufficiently small, as compared to a_{ij} , since the variable coefficient is continuous in \underline{q} and zero at $\underline{q} = 0$. The second bracketed term is also positive definite since the lead term, $m U_m$, is positive definite. Thus, the system is unstable since V_1 has variable sign in the neighborhood of the rest point and \dot{V}_1 is positive definite.

Example 12, [5] Instability Theorem

Again, consider the above conservative system, except now we assume that V is indefinite in the neighborhood of the rest point. The system can be shown to be unstable under this condition. The V - function used to prove this is

$$V_2 = -H \sum_{i=1}^n p_i q_i .$$

Assume that the potential V can be written as an m^{th} degree homogeneous form,

$$V = -U_m(q), \quad U_m(0) = 0.$$

In any region where $-H = -T + U_m$ is positive, U_m must be positive also since $-T \leq 0$. Therefore, we define the region C as that region in the neighborhood of the rest point where $\sum_{i=1}^n p_i q_i > 0$ and $-H > 0$. In region C, $V_2 > 0$ and on the boundary of region C, $V_2 = 0$.

The time derivative of V_2 is given by:

$$\dot{V}_2 = -H \left\{ \sum_{i,j=1}^n a_{ij} p_i p_j + \sum_{i,j=1}^n (A_{ij} + \sum_{k=1}^n q_k \frac{\partial A_{ij}}{\partial q_k}) p_i p_j \right\} + mH U_m,$$

where $\dot{H} = 0$. The bracketed term is non-negative and U_m and $-H$ are positive in region C. Thus, in region C, $V > 0$ and $\dot{V} > 0$. The system is unstable in the neighborhood of the rest point.

Example 13, [7] Nuclear Reactor Dynamics

This example deals with the kinetic analysis of a nuclear reactor. The knowledge of the inherent stability of the reactor and the character of its responses to suddenly induced changes in reactivity are important relative to effecting the optimal design of control systems. The system of nonlinear equations for a class of homogeneous reactors, neglecting the delayed neutron effects, is

$$\frac{d \log P}{dt} = - \frac{\alpha}{\tau} \bar{T}, \quad \alpha > 0,$$

$$\frac{dT}{dt} = \frac{1}{\epsilon} \left[(P - P_e) \right],$$

where

where

E = reactivity,

P = total power generated in the reactor > 0 ,

P_e = total power extracted from the reactor,

T = reactor temperature ($T = 0$ at the reactor's equilibrium point),

$-\alpha$ = temperature coefficient of reactivity,

τ = mean lifetime of the neutrons,

ϵ = thermal capacity.

The two operating conditions which will be considered are

(A) Constant power extraction: $P_e = P_0$,

(B) Newton's Law of Cooling: $P_e = \lambda (T - T_0)$,

where $P_0 > 0$, $\lambda > 0$, $T_0 < 0$, and T_0 is the ambient temperature of the surrounding medium.

The problem is to study the "stability conditions" of the equilibrium point $(P, T) \equiv (P_e, 0)$ for operating conditions (A) and (B). The system equations can be thought of as describing the motion of a sphere on a surface where $(-\log P)$ is the horizontal component of "displacement", $\frac{\alpha T}{\tau}$ is the corresponding "velocity" component, $\frac{\alpha \dot{T}}{\tau}$ is the corresponding "acceleration" component, $\frac{\tau \epsilon}{\alpha}$ is the "mass", and $(P - P_e)$ is the generalized "forcing" function. The candidate for the Liapunov function is the Hamiltonian or total energy of the system. The generalized potential energy is

$$\int_{\log P_{e0}}^{\log P} (P - P_e) d(\log P),$$

and the generalized kinetic energy is

$$\frac{\epsilon \alpha}{2 \tau} T^2.$$

Thus, for this conservative system, the sum of the above two terms is a constant,

$$V = \int_{\text{Log } P_{e0}}^{\text{Log } P} (P - P_e) d(\text{Log } P) + \frac{\alpha \epsilon}{2\tau} T^2 = C_0,$$

where C_0 is a constant determined by the initial conditions.

For operation (A), $P_e = P_0 = \text{constant}$. Thus, the V - function becomes

$$V = P - P_0 - P_0 \text{Log } P/P_0 + \frac{\alpha \epsilon}{2\tau} T^2 = C_0,$$

where $C_0 > 0$ in a neighborhood of $(P_0, 0)$. Since $V > 0$ and constant for all $t \geq t_0$, then V is positive definite and $\dot{V} = 0$. The system is stable in the neighborhood of $(P_0, 0)$ if $P_c > 0$ and for any T .

For operation (B), $P_e = \lambda (T - T_0)$ where $T_0 < 0$. Thus, the V - function becomes

$$V = \int_{\text{Log } P_{e0}}^{\text{Log } P} \left\{ P - \lambda T + \lambda T_0 \right\} d(\text{Log } P) + \frac{\alpha \epsilon}{2\tau} T^2 = C_0;$$

or using

$$- \lambda T d(\text{Log } P) = - \lambda T \left\{ \frac{d \text{Log } P}{dt} \right\} dt = \frac{\lambda \alpha}{\tau} T^2 dt$$

we have

$$\begin{aligned} V_1 &= V - \frac{\lambda \alpha}{\tau} \int_0^t T^2 dt = P + \lambda T_0 + \lambda T_0 \text{Log} \left\{ \frac{-P}{T_0} \right\} + \frac{\alpha \epsilon}{2\tau} T^2 \\ &= C_0 - \frac{\lambda \alpha}{\tau} \int_0^t T^2 dt. \end{aligned}$$

$$\text{Since } \left\{ P + \lambda T_0 + \lambda T_0 \text{Log} \left\{ \frac{-P}{\lambda T_0} \right\} + \frac{\alpha \epsilon}{2\tau} T^2 \right\}$$

is positive definite for $P > 0$ and all T and since $\dot{V}_1 = - \frac{\lambda \alpha}{\tau} T^2$

is negative semi-definite, the reactor is asymptotically stable in the neighborhood of the equilibrium point $(-\lambda T_0, 0)$.

Example 14, [8] Heterogeneous Reactor

In this example we consider a heterogeneous reactor. The dynamic behavior of a heterogeneous reactor with unit average power and consisting of n - media with heat generated in each medium is given by:

$$\frac{d \text{Log } P}{dt} = - \sum_{j=1}^n \alpha_j T_j, \quad \alpha_j > 0,$$

$$C_i \dot{T}_i = \eta_i (P - 1) - \sum_{j=1}^n X_{ij} (T_i - T_j),$$

$$(i = 1, \dots, n)$$

where

$$P = P(t) = \frac{\text{reactor power}}{\text{stationary power}} > 0,$$

$T_i = T_i(t)$ = deviation of the temperature from the equilibrium temperature in the i^{th} medium,

$-\alpha_i$ = $\frac{\text{temperature coefficient of reactivity}}{\text{mean life of neutrons}}$,

C_i = heat capacity of i^{th} medium,

η_i = fraction of power generated in the i^{th} medium,

X_{ij} = thermal conductivity from the i^{th} medium to the j^{th} medium.

Also, the effect of delayed neutrons is neglected; $X_{ij} = X_{ji}$; and

$$\sum_{i=1}^n \eta_i = 1. \quad \text{The null or equilibrium solution is } P = 1, T_1 = T_2 = \dots = T_n = 0.$$

The stability analysis of the null solution will be discussed for the case of two media; the analysis would be similar for the n^{th} order reactor.

The defining equations are

$$\frac{d(\text{Log } P)}{dt} = -\alpha_1 T_1 - \alpha_2 T_2,$$

$$\epsilon_1 \dot{T}_1 = \gamma_1 (P-1) - X (T_1 - T_2),$$

$$\epsilon_2 \dot{T}_2 = \gamma_2 (P-1) - X (T_2 - T_1),$$

where $X = X_{12} = X_{21}$ and $\gamma_1 + \gamma_2 = 1$. The above equations are simplified if we change the variables as given below:

$$Q = \epsilon_1 T_1 + \epsilon_2 T_2,$$

$$T = T_1 - T_2,$$

$$P = P,$$

$$\sigma_Q = \frac{\alpha_1 + \alpha_2}{\epsilon_1 + \epsilon_2} > 0,$$

$$\sigma_T = \frac{\alpha_1 \epsilon_2 - \alpha_2 \epsilon_1}{\epsilon_1 + \epsilon_2},$$

$$\sigma_0 = X \left[\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right].$$

The new system is given by:

$$\frac{d \text{Log } P}{dt} = -\sigma_T T - \sigma_Q Q,$$

$$\frac{dQ}{dt} = P - 1,$$

$$\frac{dT}{dt} = \left\{ \gamma_1 / \epsilon_1 - \gamma_2 / \epsilon_2 \right\} (P - 1) - \sigma_0 T,$$

and the null solution is

$$(P, T, Q) \equiv (1, 0, 0).$$

A Liapunov function of the above system is formed in the following way:

$$P \left\{ \frac{d \log P}{dt} \right\} \equiv P \left\{ \frac{\dot{P}}{P} \right\} \equiv \dot{P},$$

thus,

$$\begin{aligned} \frac{dV_1}{dt} &= P \frac{d(\log P)}{dt} - \frac{d(\log P)}{dt} + \sigma_Q Q \dot{Q} + \left\{ \frac{\sigma_T T}{\eta_1/\epsilon_1 - \eta_2/\epsilon_2} \right\} \dot{T} \\ &= \dot{P} - \frac{d(\log P)}{dt} + \frac{d(1/2 \sigma_Q Q^2)}{dt} + \frac{d\left\{ \frac{\sigma_T T^2}{2(\eta_1/\epsilon_1 - \eta_2/\epsilon_2)} \right\}}{dt} \\ &= \frac{d}{dt} \left\{ P - \log P + 1/2 \sigma_Q Q^2 + \frac{\sigma_T T^2}{2(\eta_1/\epsilon_1 - \eta_2/\epsilon_2)} \right\} \\ &= - \frac{\sigma_T \sigma_Q T^2}{\eta_1/\epsilon_1 - \eta_2/\epsilon_2}. \end{aligned}$$

The null solution is such that at this point $V_1 = 1$; thus, we choose $V = V_1 - 1$ as a Liapunov function. If $\eta_1/\epsilon_1 > \eta_2/\epsilon_2$ and $\alpha_1 \epsilon_2 > \alpha_2 \epsilon_1$, then V is positive definite and \dot{V} is negative semidefinite. Therefore, the reactor is locally asymptotically stable if

$$(1) \eta_1/\epsilon_1 > \eta_2/\epsilon_2,$$

$$(2) \alpha_1 \epsilon_2 > \alpha_2 \epsilon_1.$$

Example 15, [9] Homogeneous Reactor

This example is a simple stability problem of a homogeneous reactor where delayed neutron action is considered. The defining equations are

$$\dot{n} = \frac{k - \beta}{l} n + \lambda c,$$

$$\dot{c} = \frac{\beta}{l} n - \lambda c,$$

$$\dot{T} = an - gT,$$

$$K = K_0 - rT, \quad r > 0,$$

where

n = neutron density,

c = delayed neutron precursor concentration,

T = reactor temperature,

K = reactivity,

β, l, a, r = physical constants.

Since n and C are positive or zero, a candidate for the Liapunov function is

$$V = n + c + \left(\frac{r}{2al} \right) T^2,$$

where

$$\dot{V} = \frac{K_0 n}{l} - \frac{grT^2}{al}$$

Thus, if $K_0 \leq 0$, all solutions must eventually reach the origin in the nCT - space; thus, the reactor eventually shuts off.

Example 16, [10] "Newton Law of Cooling" Reactor

The kinetic equations for a Newton Law of Cooling reactor model are

$$\dot{n} = \frac{K}{l} n,$$

$$\dot{T} = an - gT,$$

$$K = K_0 - F_1(T) - F_2(T),$$

$$F_1(T) = \text{odd function of } T,$$

$$F_2(T) = \text{even function of } T,$$

$$F_2(0) = 0, a > 0, g > 0, l > 0$$

$$|F_1(T)| + |F_2(T)| \neq \text{constant}.$$

Analyzing the stability of the equilibrium solution, $(n, T) = (0, 0)$, gives, "in the small", the following:

- (1) for $K_0 < 0$, the system is asymptotically stable in the small,
- (2) for $K_0 > 0$, the system is unstable.

If the nonlinear terms are now considered, Liapunov theory produces the following results.

(1) Choose

$$V_1 = \frac{1}{2} an + \int_0^T \{F_1(T) - F_2(T)\} dt,$$

where

$$\dot{V}_1 = anK_0 - g \{F_1(T) + F_2(T)\} T;$$

resulting in the following,

$n = T = 0$ is asymptotically stable for all initial conditions when $K_0 < 0$ and $\{F_1(t) + F_2(T)\} T \geq 0$ for all T . Also, if $K > 0$ for any temperature, or if $K_0 > 0$, and if $\{F_1(T) + F_2(T)\} T \leq 0$ in the neighborhood of the origin along the T -axis, then $n = T = 0$ is unstable by Liapunov theory.

(2) Choose

$$V_2 = \frac{1}{2} an + \int_0^T F_1(T) dT,$$

where

$$\dot{V}_2 = anK_0 - aF_2(T) n - g F_1(T) T;$$

resulting in the following,

$n = T = 0$ is asymptotically stable for all initial conditions when $K_0 < 0$ and $F_1(T) T \geq 0$ and $F_2(T) \geq 0$ for all T . Also, if $K > 0$ for any temperature, or if $K_0 > 0$, and if $F_1(T) T \leq 0$ and $F_2(T) \leq 0$ in the neighborhood of

the origin along the T - axis, then the "shut-down" solution is unstable.

There are other examples of Liapunov theory being used in the reactor-field, such as, references [13, 14, 15] .

Example 17, [11,12] Continuous Medium Reactor

This example considers the dynamic stability of a continuous medium nuclear reactor. The Liapunov theory of ordinary differential equations is not directly applicable but the authors use a Hamiltonian, a "Liapunov-like" function, in their analysis; thus, their example is presented here.

The physical assumptions considered in their problem are listed in reference [12] . The problem is the continuum extension of the n-distinct media problem considered in a previous example. The governing equations are

$$\frac{d \text{Log } P}{dt} = - \int \alpha(x) T(x, t) dx,$$

$$\frac{dT(x, t)}{dt} = \eta(x) \{ P(t) - 1 \} + X \frac{\partial^2 T}{\partial x^2},$$

where the thermal conductivity, X, satisfies

$$0 < X \equiv \text{constant} < < 1,$$

$$\alpha(x) \geq 0,$$

$$\eta(x) \geq 0,$$

$$P = \text{power of the reactor},$$

$$T = \text{temperature},$$

and the functions $\alpha(x)$ and $\eta(x)$ are of the same order; this means physically that the heat is generated predominantly at locations where the local negative temperature coefficient is large. The equilibrium solution is $(P, T) = (1, 0)$. Thus, the problem is to find the sufficient conditions

for damping the power oscillations of the reactor in the neighborhood of (1, 0).

Because $0 < X \ll 1$, the first approximation T_1 and P_1 for T and P must satisfy

$$T_1 = \eta(x) \int_0^t \{P_1 - 1\} dt.$$

For this approximation the motion about the equilibrium point can be characterized by a "Hamiltonian" or a "Liapunov function", H_1 , which is constant in time and is defined by

$$H_1 = \frac{1}{2} \left\{ \frac{d \log P_1}{dt} \right\}^2 + \int \alpha(x) \eta(x) \{P_1 - \log P_1\} dx,$$

where the first term on the right is a generalized kinetic energy and the second term a generalized potential energy. From the first "system" equation and the expression for T_1 , the time derivative of H_1 is shown to be zero.

The second approximation is obtained by substituting T_1 into $X \frac{\partial^2 T}{\partial x^2}$ of the second "system" equation; thus,

$$X \frac{\partial^2 T}{\partial x^2} = X \int_0^t \eta''(x) \{P_1 - 1\} dt.$$

The Hamiltonian, H_2 , is constructed in an analogous manner and is positive definite. The time derivative of H_2 is

$$\dot{H}_2 = - \left\{ X \frac{d \log P_1}{dt} \int_0^t (P_1 - 1) dt \right\} \int_0^x \alpha(x) \eta''(x) dx,$$

where the term in $\left\{ \text{---} \right\}$ is inherently negative because of the first "system" equation. The integral in \dot{H}_2 can be written as:

$$\int_0^x \alpha(x) \eta''(x) dx = \lim_{h \rightarrow 0} \frac{1}{h} \left\{ \int_0^x \alpha(x) \eta(x+h) dx + \int_0^x \alpha(x) \eta(x-h) dx - 2 \int_0^x \alpha(x) \eta(x) dx \right\},$$

where the final integral on the right dominates the sum of the other two integrals because $\alpha(x)$ and $\eta(x)$ are of similar ordering. Thus,

$$\int_0^x \alpha(x) \eta''(x) dx \leq 0 \text{ and } \dot{H}_2 \leq 0,$$

which implies that the oscillations of the reactor about (1, 0) are nonincreasing.

Example 18, [5] Motion of a Projectile

The stability of the rotational motion of a projectile is analyzed by Chetaev's method of linear combination of first integrals. Let θ be the angle which the axis of the projectile forms with its projection upon the vertical plane of the line of fire. The angle α is measured between the above projection and the trajectory of the center of gravity. It is assumed that the center of gravity moves linearly and uniformly. Other terms in the equations of motion are:

C = polar moment of inertia

A = moment of inertia about transverse axis through the center of gravity

n = projection of angular velocity

e = distance from c.g. to center of pressure

R = forward resistance.

The equations of motion are

$$\begin{aligned} A \ddot{\theta} + A \dot{\alpha}^2 \sin \theta \cos \theta - C_n \dot{\alpha} \cos \theta &= e R \sin \theta \cos \alpha, \\ A \ddot{\alpha} \cos \theta - 2A \dot{\alpha} \dot{\theta} \sin \theta + C_n \dot{\theta} &= e R \sin \alpha. \end{aligned}$$

The equilibrium solution under consideration is $\alpha = \dot{\alpha} = \theta = \dot{\theta} = 0$.

Thus, the above equations can be considered the equations of distributed motion about the equilibrium position.

The first integrals are the energy integral,

$$F_1 = \frac{A}{2} (\dot{\theta}^2 + \dot{\alpha}^2 \cos^2 \theta) + eR (\cos \alpha \cos \theta - 1),$$

and the momentum integral,

$$\begin{aligned} F_2 &= A (\dot{\theta} \sin \alpha - \dot{\alpha} \cos \theta \sin \theta \cos \alpha) + \\ &+ C_n (\cos \alpha \cos \theta - 1). \end{aligned}$$

Both $\frac{dF_1}{dt}$ and $\frac{dF_2}{dt}$ can be shown to be zero; thus F_1 and F_2 are constants,

that is, first integrals. Neither F_1 nor F_2 is definite with respect to sign. We now form a new first integral:

$$\begin{aligned} V &= F_1 - \lambda F_2 \\ &= 1/2 \left\{ A \dot{\alpha}^2 + 2A \lambda \dot{\alpha} \theta + (C_n \lambda - eR) \theta^2 \right\} + \\ &+ 1/2 \left\{ A \dot{\theta}^2 - 2A \lambda \dot{\theta} \alpha + (C_n \lambda - eR) \alpha^2 \right\} + \\ &+ \left\{ \text{terms no lower than 3rd order} \right\}. \end{aligned}$$

V , locally, is positive definite if the quadratic forms are positive definite.

Therefore, for stability we require that

$$C_n^2 - 4 A e R > 0$$

which gives the lower limit of the angle of rotational motion of the projectile such that the axis will follow the tangent to the trajectory of the center of gravity. We have unstable motion if

$$C_n^2 - 4 A e R < 0.$$

Example 19, [16] Liquid-Filled Gyroscope

In this example, the necessary and sufficient conditions for stability are derived for the motion of a gyroscope containing an ellipsoidal cavity filled with an ideal, incompressible liquid. The terms involved in this problem are:

$L_1, L_2, 0$ = Components of the angular momentum of the moment of the gravity forces,

p, q, r = Components of the instantaneous angular velocity of the gyroscope,

P_1, q_1, r_1 = Components of $1/2 \text{ rot } \underline{V}$, where \underline{V} is the rectilinear velocity,

A_1, C_1, A_2, C_2 = Moments of inertia of the liquid and solids.

ϵ = Eccentricity of the ellipsoid,

$a=b, c$ are semi-axes, and $A_2 = (1 - \epsilon) C_2$,

M = Mass of the liquid in the cavity,

P = Weight of the system,

h = Distance from fixed point to c.g.,

$\gamma_1, \gamma_2, \gamma_3$ = Direction cosines of the line of action of the weight vector in the given coordinate system,

where

$$L_1 = Ph \gamma_2 \text{ and } L_2 = - Ph \gamma_1 .$$

The equations of motion are

$$\dot{p}_1 = r_1 \left\{ \epsilon q_1 - (1 + \epsilon) q \right\},$$

$$\dot{q}_1 = -r_1 \left\{ \epsilon p_1 - (1 + \epsilon) p \right\},$$

$$\dot{r}_1 = (1 - \epsilon) (p_1 q - q_1 p),$$

$$A_1 \dot{p} + A_2 \dot{p}_1 + q (C_1 r + C_2 r_1) = L_1,$$

$$A_1 \dot{q} + A_2 \dot{q}_1 - p (C_1 r + C_2 r_1) = L_2,$$

$$C_1 \dot{r} + C_2 \dot{r}_1 - A_2 (p_1 q - q_1 p) = 0,$$

$$\dot{\gamma}_1 = -q \gamma_3,$$

$$\dot{\gamma}_2 = p \gamma_3,$$

$$\dot{\gamma}_3 = q \gamma_1 - p \gamma_2.$$

For the case of a very thin shell, the shell's moments of inertia are neglected.

Thus, assuming that the fixed point is the c.g. and that

$$A_1 = \frac{M}{5} (a^2 + c^2)^2, \quad C_1 = 0,$$

$$C_2 = \frac{2}{5} M a^2, \quad A_2 = (1 - \epsilon) \epsilon^2,$$

the middle three equations, above, reduce to

$$p = - \left\{ \frac{1 + \epsilon}{\epsilon} \right\} r_1 \left\{ q_1 (1 - \epsilon) + \epsilon q \right\},$$

$$q = \left\{ \frac{1 + \epsilon}{\epsilon} \right\} r_1 \left\{ p_1 (1 - \epsilon) + \epsilon p \right\}.$$

The first integrals of the system are

The first integrals of the system are

$$F_1 = p^2 + q^2 - \frac{1 + \epsilon}{\epsilon} r_1^2,$$

$$F_2 = p_1^2 + q_1^2 + \frac{1 + \epsilon}{1 - \epsilon} r_1^2,$$

$$F_3 = 2pp_1 + 2qq_1 + \frac{1 + 2\epsilon}{1 - \epsilon} r_1^2.$$

Eliminating r_1 gives

$$V_1 = p^2 + q^2 + \frac{1 - \epsilon}{\epsilon} (p_1^2 + q_1^2),$$

$$V_2 = 2pp_1 + 2qq_1 - \frac{1 + 2\epsilon}{1 + \epsilon} (p_1^2 + q_1^2).$$

The linear combination

$$V = V_1 + \lambda V_2$$

is positive definite if

$$\frac{4 + 5\epsilon}{\epsilon(1 + \epsilon)^2} > 0,$$

and $\dot{V} = 0$. This gives the necessary and sufficient conditions for stability, with respect to the variables p, q, p_1, q_1 , of the motion of the gyroscope about its vertical position of equilibrium. These conditions are given as

$$(1) \epsilon > 0, (a^2 > c^2),$$

or

$$(2) \epsilon < -4/5, (c^2 > 9a^2).$$

Example 20, [17] Gyroscope on Gimbals

This example considers the sufficient conditions for stability of motion of the regular precession of a gyroscope on gimbals. There is assumed to be no frictional forces on the gimbal axes, only gravitational forces are present. The important terms are

Θ = angle of nutation,

χ = angle of precession,

ϕ = angle of rotation of the gyroscope,

A, B, C = principle moments of inertia of the gyroscope,

A_1, B_1, C_1 = principle moments of inertia of the inner gimbal,

A_2 = moment of inertia of the outer ring

p = weight of the gyroscope and the inner ring

$(0, 0, z_0)$ = coordinates of the center of gravity.

The equations of motion are

$$C \frac{d}{dt} (\dot{\phi} + \dot{\psi} \cos \Theta) = 0,$$

$$(A + A_1) \ddot{\Theta} - (A + B_1 - C_1) \dot{\psi}^2 \sin \Theta \cos \Theta + C (\dot{\phi} + \dot{\psi} \cos \Theta) \dot{\psi} \sin \Theta - p z_0 \sin \Theta = 0,$$

$$\frac{d}{dt} \left\{ (A + B_1) \dot{\psi}^2 \sin^2 \Theta + C (\dot{\phi} + \dot{\psi} \cos \Theta) \cos^2 \Theta + C_1 \dot{\psi}^2 \cos^2 \Theta + A_2 \dot{\psi} \right\} = 0.$$

The first integrals for this system are

$$K = (A + B_1) \dot{\psi}^2 \sin^2 \Theta + C (\dot{\phi} + \dot{\psi} \cos \Theta) \cos \Theta + C_1 \dot{\psi}^2 \cos^2 \Theta + A_2 \dot{\psi},$$

$$r_0 = \dot{\phi} + \dot{\psi} \cos \Theta,$$

$$h = (A + A_1) \dot{\Theta}^2 + (A + B_1) \dot{\psi}^2 \sin^2 \Theta + C_1 \dot{\psi}^2 \cos^2 \Theta + C (\dot{\phi} + \dot{\psi} \cos \Theta)^2 + A_2 \dot{\psi}^2 + 2p z_0 \cos \Theta.$$

Consider the equilibrium condition defined by:

$$\Theta = \Theta_0 = \text{constant}$$

$$\dot{\Theta} = 0$$

$$\dot{\psi} = \nu = \text{constant}$$

$$r = \omega = \dot{\phi} + \dot{\psi} \cos \Theta = \text{constant}$$

where

$$\left\{ (A + B_1 - C_1) \nu^2 \cos \Theta_0 - C \omega \nu + p z_0 \right\} \sin \Theta_0 = 0.$$

For regular precession, Θ_0 is not equal to 0 or π .

We now apply the following transformation to the equations of motion:

$$\begin{aligned}\Theta &= \Theta_0 + \eta, \\ \dot{\Theta} &= \dot{\eta} = \xi_1, \\ \dot{\chi} &= \nu + \xi_2, \\ r &= \omega + \xi_3.\end{aligned}$$

The first integrals corresponding to the new coordinates,

$(\eta, \xi_1, \xi_2, \xi_3)$, are

$$\begin{aligned}V_1 &= (A + A_1)\xi_1^2 + \{D\nu - pZ_0\cos\Theta_0\}\eta^2 + E(\xi_2^2 + 2\nu\xi_2) + \\ &+ 2F\nu\xi_2\eta + C(\xi_3^2 + 2\omega\xi_3) + F\nu^2\eta + \\ &- pZ_0(\sin\Theta_0)\eta + \text{higher order terms},\end{aligned}$$

$$\begin{aligned}V_2 &= \left\{D - \frac{C\omega}{2}\cos\Theta_0\right\}\eta^2 + F\nu\eta - C\omega(\sin\Theta_0)\eta + \\ &+ E\xi_2 + F\xi_2\eta + C(\cos\Theta_0)\xi_2 - C(\sin\Theta_0)\eta\xi_2 + \\ &+ \text{higher order terms},\end{aligned}$$

+ higher order terms,

$$V_3 = \xi_3 = \text{constant},$$

where

$$D = (A + B_1 - C_1)\nu(\cos^2\Theta_0 - \sin^2\Theta_0),$$

$$E = (A + B_1)\sin^2\Theta_0 + C_1\cos^2\Theta_0 + A_2,$$

$$F = (A + B_1 - C_1)\cos\Theta_0\sin\Theta_0.$$

A candidate for a Liapunov function is

$$V = V_1 - 2\nu V_2 + 2C(\nu\cos\Theta_0 - \omega)V_3 + \frac{C^2}{A + B_1 - C_1} V_3^2,$$

where $\dot{V} = 0$. V is positive definite if

$$D\nu - C\omega\nu\cos\Theta_0 + pZ_0\cos\Theta_0 < 0;$$

this is the sufficient condition for stability of the regular precession.

If we now use the equation given above, namely,

$$F \mathcal{N}^2 - c \omega \mathcal{N} \sin \Theta_0 + p \mathcal{Z}_0 \sin \Theta_0 = 0,$$

to define Θ_0 , the stability condition for the regular precession of a gyroscope on gimbals is

$$A + B_1 - C_1 > 0.$$

This stability is with reference to Θ , $\dot{\Theta}$, $\dot{\gamma}$ and r .

It can be shown from the above Liapunov function that if $\Theta_0 = 0$, the necessary and sufficient condition for stability of the unperturbed motion is

$$(A + B_1 - C_1) \mathcal{N}^2 - c \omega \mathcal{N} + p \mathcal{Z}_0 < 0.$$

The unperturbed motion in this case is the uniform rotation of the outer ring about a vertical axis with angular velocity \mathcal{N} and a uniform rotation of the gyroscope with angular velocity ω .

Consider the above case, $\Theta_0 = 0$, with the friction of the gimbals being taken into consideration. The equilibrium solution is

$$\Theta = 0, \dot{\Theta} = 0, \dot{\gamma} = \mathcal{N}, \dot{\phi} = \omega_1 = \omega - \mathcal{N}.$$

The transformation equations are

$$\begin{aligned} \Theta &= \eta, \dot{\Theta} = \dot{\eta} = \xi_1, \dot{\gamma} = \mathcal{N} + \xi_2, \\ \dot{\phi} &= \omega_1 + \xi_3; \end{aligned}$$

and the Rayleigh dissipation function is

$$2f^* = a \xi_1^2 + b \xi_2^2 + c \xi_3^2 + 2e \xi_2 \xi_3 + 2f \xi_3 \xi_1 + 2g \xi_1 \xi_2,$$

where the constants a , b , c , e , f , and g are such that f^* is positive definite in ξ_1 , ξ_2 , ξ_3 . Thus, the variational equations for the perturbed motion are

$$(A + A_1) \dot{\xi}_1 - [(A + B_1 - C_1) \mathcal{N}^2 - C \omega \mathcal{N} + p z_0] \eta = - (a \dot{\xi}_1 + g \dot{\xi}_2 + f \dot{\xi}),$$

$$(A_2 + C + C_1) \dot{\xi}_2 + C \dot{\xi} = - (g \dot{\xi}_1 + b \dot{\xi}_2 + e \dot{\xi}),$$

$$C(\dot{\xi} + \dot{\xi}_2) = - (f \dot{\xi}_1 + e \dot{\xi}_2 + C \dot{\xi}).$$

We now consider a Liapunov function of the same form as before:

$$2W = (A + A_1) \dot{\xi}_1^2 + (A_2 + C + C_1) \dot{\xi}_2^2 - [(A + B_1 - C_1) \mathcal{N}^2 - C \omega \mathcal{N} + p z_0] \eta^2 + \epsilon \dot{\xi}^2 + 2c \dot{\xi}_2 \dot{\xi} + 2 \epsilon (A + A_1) \eta \dot{\xi}_1,$$

where $\epsilon > 0$, and

$$\dot{W} = - \left\{ [a - \epsilon (A + A_1)] \dot{\xi}_1^2 + b \dot{\xi}_2^2 + c \dot{\xi}^2 + 2e \dot{\xi}_2 \dot{\xi} + 2f \dot{\xi}_1 \dot{\xi} + 2g \dot{\xi}_1 \dot{\xi}_2 - \epsilon [(A + B_1 - C_1) \mathcal{N}^2 - C \omega \mathcal{N} + p z_0] \eta^2 + \epsilon \eta (a \dot{\xi}_1 + g \dot{\xi}_2 + f \dot{\xi}) \right\}.$$

Thus, $W > 0$ and $\dot{W} < 0$ if

$$(A + B_1 - C_1) \mathcal{N}^2 - C \omega \mathcal{N} + p z_0 < \delta(\epsilon) < 0$$

for sufficiently small ϵ . Then the motion, $\Theta = 0$, $\dot{\Theta} = 0$,

$\dot{\psi} = \mathcal{N}$ and $\dot{\phi} = \omega - \mathcal{N}$ which is stable without friction becomes

asymptotically stable when dissipation forces are present.

Example 21, [18] Motion of a Tippe - Top

This example considers the stability of a "tippe - top"; that is, a top with a spherical base whose center of gravity is below the center of curvature. The important terms are

p, q, r = projection of the instantaneous value of the angular velocity on the moving coordinate system,

$\gamma_1, \gamma_2, \gamma_3$ = direction cosines of the direction of the force of gravity, mg ,

A, B, C = principal moments of inertia of the top,

\underline{L}_0 = angular momentum of the top about the c.g.,

$\frac{d\underline{r}}{dt}$ = velocity of c.g. = (u, v, w) ,

a = radius of the sphere,

l = center of the sphere to c.g.

The equations of motion are

$$A\dot{p} + (C - A)qr + ma\gamma_2\dot{w} - m(a\gamma_3 - l)\dot{v} = mg l \gamma_2,$$

$$A\dot{q} + (A - C)pr + m(a\gamma_3 - l)\dot{u} - ma\gamma_1\dot{w} = -mg l \gamma_1,$$

$$C\dot{r} + ma\gamma_1\dot{v} - ma\gamma_2\dot{u} = 0,$$

$$w = a(\gamma_2 p - \gamma_1 q),$$

$$u = a\gamma_3 - l\gamma_1 g - a\gamma_2 r,$$

$$v = a\gamma_1 r - (a\gamma_3 - l)p,$$

$$\dot{\gamma}_1 = r\gamma_2 - q\gamma_3,$$

$$\dot{\gamma}_2 = p\gamma_3 - r\gamma_1,$$

$$\dot{\gamma}_3 = q\gamma_1 - p\gamma_2.$$

From these equations there results one energy integral, two momentum integrals, and the relationship

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

Our problem is to consider the stability of rotation of the top about its

vertical axis; that is, the unperturbed motion is

$$u = v = w = 0,$$

$$p = q = 0,$$

$$r = r_0 = \text{constant},$$

$$\gamma_1 = \gamma_2 = 0,$$

$$\gamma_3 = 1.$$

The transformation equations giving the perturbed motion are

$$p = \xi_1, q = \xi_2, r = r_0 + \xi_3, \gamma_1 = \eta_1,$$

$$\gamma_2 = \eta_2, \gamma_3 = 1 + \eta_3, \omega_1 = r_0 + \xi_4,$$

or

$$\xi_4 = \xi_1 \eta_1 + \xi_2 \eta_2 + \xi_3 \eta_3 + r_0 \eta_3 + \xi_3.$$

From the perturbed equations, the first integrals V_1, V_2, V_3, V_4 are formed:

$$V_1 = \left\{ A + m \left[a^2 - 2a l (1 + \eta_3) + l^2 \right] \right\} (\xi_1^2 + \xi_2^2) + \\ + 2 m g l \eta_3 + 2 m a l \xi_4 (r_0 + \xi_3) + \left\{ C + m a \left[a + \right. \right. \\ \left. \left. - l (1 + 2 \eta_3) \right] \right\} (2 r_0 \xi_3 + \xi_3^2) - m a^2 \xi_4 (2 r_0 + \xi_4).$$

$$V_2 = A a \xi_4 + (C - A) a r_0 \eta_3 + [C(a - l) - A a] \xi_3 + \\ + (C - A) a \xi_3 \eta_3,$$

$$V_3 = [(C - A) (C + m a^2) - C m l^2] \xi_3^2 +$$

$$+ 2 r_0 [(C - A) (C + m a^2) - C m l^2] \xi_3 - 2 \xi_4 m a r_0 [C(a - l) + \\ - A a] + A m a^2 \xi_4^2,$$

$$V_4 = \eta_1^2 + \eta_2^2 + 2 \eta_3 + \eta_3^2 = 0.$$

The choice for a Liapunov function is

$$V = V_1 + \lambda V_2 + \mu V_3 + \nu V_4 + \rho V_4^2,$$

where if

$$\lambda = \frac{2(mg l + \nu) [C - A + m l (a - l)] - m a r_o^2 [a(A - C) + A l - m l^2 (a - l)]}{a r_o^2 (A - C) [C + m a (a - l)]}$$

and

$$\mu = \frac{C a r_o^2 - (a - l) [m l (g + a r_o^2) + \nu]}{a r_o^2 (A - C) [C + m a (A - l)]},$$

The linear terms in V drop out. V is positive definite and thus the top is stable about the vertical axis if

- (1) $A_2 > 0$,
- (2) $4 \nu A_1 - A_5^2 > 0$,
- (3) $A_2 A_4 - A_3^2 > 0$,
- (4) $\nu > 0$,
- (5) $A_4 > 0$,

where

$$A_1 = A + m(a - l)^2,$$

$$A_2 = C + 2 m a^2 + \mu C [C - A + m (a^2 - l^2)],$$

$$2A_3 = C a [\lambda - 2 \mu m r_o (a - l)],$$

$$A_4 = r_o^2 m a^2 (1 + \mu A) + \nu + 4 \rho,$$

$$A_5 = a [-2 m r_o (a - l) (1 + \mu C) + A (2 \mu m r_o a + \lambda)].$$

Example 22, [19] Motion of a Gyrostat

The problem of a gyrostat consisting of the rigid body T_1 and the rotors T_2 , whose axes are fixed in T_1 , is now considered. The particular problem being analyzed is a free gyrostat in the Newtonian force field. Let O be the origin of a fixed Cartesian coordinate system, ξ, η , and ζ , coinciding with the center of gravitation attraction. The gyrostat moves in a Newtonian central gravitational field, and the axes of the moving coordinate system x, y and z coincides with the principal central axes of inertia of the gyrostat. The terms used in this problem are:

A, B, C = principal central moments of inertia

M = mass of gyrostat

T = combined system T_1 and T_2

$R = (\xi^2 + \eta^2 + \zeta^2)^{\frac{1}{2}}$ = distance to the center of mass of T

τ_1, τ_2, τ_3 = direction cosines of the radius vector

k_1, k_2, k_3 = components of angular momentum of T_2

p, q, r = components of angular velocity of T_1

$\gamma_1, \gamma_2, \gamma_3$ = direction cosines between xyz - system and ξ -axis

U = Newtonian potential.

For a free gyrostat we have $A = C, k_1 = k_3 = 0, k_2 = k(t)$, where $k(t)$ is a bounded continuous function of time. Thus, the equation for U and the motion equations are given by

$$U = \left\{ \frac{4\pi M}{R} - \frac{3A}{2R} \left[(B - A) \tau_2^2 - \frac{B - A}{3} \right] \right\},$$

$$M \ddot{\xi} = \frac{\partial U}{\partial \xi}, \quad M \ddot{\eta} = \frac{\partial U}{\partial \eta}, \quad M \ddot{\zeta} = \frac{\partial U}{\partial \zeta},$$

$$A \dot{p} + (A - B) q r - r k(t) = \frac{3H}{R^3} (A - B) \tau_3 \tau_2,$$

$$B \dot{q} + k(t) = 0$$

$$A \dot{r} + (B - A) p q + p k(t) = \frac{3H}{R^3} (B - A) \tau_2 \tau_1.$$

The first integrals are

$$B q + k(t) = H = \text{constant},$$

$$M \left[\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 \right] + A (p^2 + r^2) - 2U = \text{constant},$$

which is the energy integral, and the next equation is the angular momentum integral,

$$M \left[\xi \dot{\eta} - \eta \dot{\xi} \right] + A p \gamma_1 + (B q + k(t)) \gamma_2 + A r \gamma_3 = \text{constant}.$$

Further, we have the trivial relationships

$$\tau_1^2 + \tau_2^2 + \tau_3^2 = 1,$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

The introduction of spherical coordinates, whose origin coincides with the center of mass of the system, admits the following form for the particular solution of the equations of motion:

$$\begin{aligned} p = r = 0, \quad q &= B^{-1} (H - k(t)), \\ \gamma_1 = \gamma_3 = 0, \quad \gamma_2 &= 1, \quad R = R_0, \quad \dot{R} = 0, \\ \chi = 0, \quad \dot{\chi} &= 0, \quad \phi = (\omega t + \phi_0), \quad \dot{\phi} = \omega, \\ \tau_2 = 0, \quad \tau_1 &= \sin \nu(t), \quad \tau_3 = \cos \nu(t), \\ \dot{\nu} &= \omega - q(t). \end{aligned}$$

These equations describe the motion of the center of mass on the circular orbit, with radius R_0 and with constant angular velocity ω ; and they describe the

rotation of the gyrostat about its axis of symmetry with angular velocity q , while the rotor performs a prescribed motion such that $Bq + k(t) = H$.

We now determine the Liapunov stability of this non-perturbed motion.

First we change the variables by applying the following relationships to the equations of motion:

$$\gamma_2 = 1 + x_2 \quad R = R_0 + x_3, \quad \dot{\phi} = \omega + \dot{x}_4.$$

From the transformed equations we get the following first integrals

$$\begin{aligned} V_1 = & MR_0^2 \dot{\gamma}^2 + M \dot{x}_3^2 - MR_0^2 \omega^2 \gamma^2 + \\ & + \left[2 MR_0 \omega^2 + \frac{2HM}{R_0^2} + \frac{6(B-A)H}{R_0^4} \right] x_3 + \\ & + \left[M \omega^2 - \frac{2HM}{R_0^3} - \frac{6(B-A)H}{R_0^5} \right] x_3^2 + \\ & + 4MR_0 \omega x_3 \dot{x}_4 + MR_0^2 \dot{x}_4^2 + 2MR_0 \dot{x}_4 + \frac{34(B-A)H^2}{R_0^3} + \\ & + A(p^2 + r^2) + \text{higher order terms} = \text{constant}, \end{aligned}$$

$$\begin{aligned} V_2 = & MR_0^2 \dot{x}_4 + 2MR_0 \omega x_3 + 2MR_0 x_3 \dot{x}_4 + M \omega x_3^2 + \\ & - MR_0^2 \omega \gamma^2 + A(p \gamma_1 + r \gamma_3) + H x_2 + x_1 + \\ & + x_1 x_2 + \text{higher order terms} = \text{constant}, \end{aligned}$$

$$V_3 = x_1 = \text{constant}, \quad V_4 = \gamma_1^2 + \gamma_3^2 + x_2^2 + 2x_2.$$

The stability of the non-perturbed motion is investigated by constructing a Liapunov function by Chetaev's method. The candidate for a Liapunov function is

$$W = V_1 - 2\omega(V_2 - V_3) + H \omega V_4 + \lambda_1 V_2^2 + \lambda_2 V_3^2,$$

where λ_1 and λ_2 are constants. W is a positive definite form if

$$B > A, H > A\omega,$$

and if all principal diagonal minors of the determinant of C_{ij} are positive:

$$C_{ij} = C_{ji} \quad i, j = 1, 2, 3, 4,$$

$$C_{11} = \lambda_1 M R_0^2 \omega^2 - 3 M \omega^2 - \frac{3(B-A)H}{R_0^5},$$

$$C_{12} = 2 \lambda_1 M^2 R_0^3 \omega, \quad C_{13} = 2 \lambda_1 M R_0 H \omega,$$

$$C_{14} = 2 \lambda_1 M R_0 \omega, \quad C_{22} = M R_0^2 + \lambda_1 M^2 R_0^4,$$

$$C_{23} = \lambda_1 M R_0^2 H, \quad C_{24} = \lambda_1 M R_0^2, \quad C_{33} = H \omega + \lambda_1 H^2,$$

$$C_{34} = -\omega + \lambda_1 H, \quad C_{44} = \lambda_1 + \lambda_2.$$

In most practical cases λ_1 and λ_2 can be so chosen such that the conditions on C_{ij} are satisfied. Thus, the non-perturbed motion of a gyrostat with one rotor, whose angular momentum satisfies the condition

$$Bq + K(t) - A\omega > 0,$$

is stable.

Example 23, [20] Liquid-Filled Rockets

This problem is concerned with the stability studies of continuous media with respect to a finite number of parameters which describe the motion through integral relationships. Examples of these parameters could be the coordinates of the center of gravity of a bounded volume of continuous medium, or the projections of the linear momentum of the medium on certain axes. The time variation of these parameters is described by ordinary differential equations. The stability of the motion of a continuous medium with respect

to the above mentioned parameters will be called the conditional stability of the motion of a continuous medium.

The particular example being considered is the stability of motion of a rotating solid, with a liquid-filled cavity, with respect to parameters describing the motion of the solid and the projections of the angular momentum of the liquid. This solid is a free solid with a completely or partially liquid-filled cavity. The liquid is ideal, non-compressible and homogeneous. Also, the central ellipsoid of inertia of the solid is an ellipsoid of revolution ($A = B, C$), and the cavity is a body of revolution whose axis coincides with the axis of the ellipsoid. If the liquid has a free surface, the pressure at the surface is assumed constant. The liquid is such that its velocity and pressure are continuous functions.

In stability problems dealing with liquid-filled bodies, we are interested mainly in the question of the stability of the motion of the solid body. The question of the stability of the liquid is only important in so far as it effects the body as a whole. In this connection it is natural to consider the question of the stability of motion of our system relative to all variables which characterize the motion of the solid body and the influence on this motion due to the motion of the liquid. This leads to the conditional stability mentioned above; that is, the stability relative to certain ones of the variables but not to all of them that determine the motion of this mechanical system. (There are an infinite number of variables because the liquid is a continuum.)

The terms used in this problem are

$O_1 X_1 Y_1 Z_1$ = fixed axes; O , at the center of mass

$O X Y Z$ = axes moving with the solid

$\gamma_1, \gamma_2, \gamma_3$ = direction cosines

T_1 = kinetic energy of the solid

T_2 = kinetic energy of the liquid

v_1, v_2, v_3 = velocity of point O

$\omega_1, \omega_2, \omega_3$ = angular velocity

M_1 = mass of the solid

v_x, v_y, v_z = velocity of fluid particles

τ = volume of the liquid

g_1, g_2, g_3 = momentum of the liquid

ρ = density of the fluid

L_1, L_2, L_3 = moment due to air pressure

$U = - a \gamma_3$ = force function of the air pressure.

It is assumed that the center of mass of the whole system is in rectilinear motion with constant velocity; this is the well-known approximation to a small segment of the flat trajectory of a missile. For this missile it is assumed that only the overturning moment of the forces of air pressure act.

The equations of motion of the system 21 relative to the center of mass are

$$A\dot{\omega}_1 + \dot{g}_1 + (C - A)\omega_2\omega_3 + \omega_2 g_3 - \omega_3 g_2 = a \gamma_2,$$

$$A\dot{\omega}_2 + \dot{g}_2 + (A - C)\omega_3\omega_1 + \omega_3 g_1 - \omega_1 g_3 = -a \gamma_1,$$

$$C\dot{\omega}_3 + \dot{g}_3 + \omega_1 g_2 - \omega_2 g_1 = 0,$$

$$\frac{d}{dt} \left[v_x + v_1 + \omega_2 z - \omega_3 y \right] + \omega_2 \left[v_z + v_3 + \omega_1 y - \omega_1 x \right]$$

$$- \omega_3 \left[v_y + v_2 + \omega_3 x - \omega_1 z \right] = - \frac{1}{e} \frac{\partial p}{\partial x},$$

$$\frac{d}{dt} \left[v_y + v_2 + \omega_3 x - \omega_1 z \right] + \omega_3 \left[v_x + v_1 + \omega_2 z - \omega_3 y \right]$$

$$- \omega_1 \left[v_z + v_3 + \omega_1 y - \omega_2 x \right] = - \frac{1}{e} \frac{\partial p}{\partial y},$$

$$\frac{d}{dt} \left[v_z + v_3 + \omega_1 y - \omega_2 x \right] + \omega_1 \left[v_y + v_2 + \omega_3 x - \omega_1 z \right]$$

$$- \omega_2 \left[v_x + v_1 + \omega_2 z - \omega_3 y \right] = - \frac{1}{e} \frac{\partial p}{\partial z},$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0,$$

$$\dot{\gamma}_1 = \omega_3 \gamma_2 - \omega_2 \gamma_3,$$

$$\dot{\gamma}_2 = \omega_1 \gamma_3 - \omega_3 \gamma_1,$$

$$\dot{\gamma}_3 = \omega_2 \gamma_1 - \omega_1 \gamma_2.$$

Applying the general theorems on relative motion of a mechanical system about its mass center, we obtain some of the first integrals of the equations of motion of a solid with liquid in its cavity. The total kinetic energy of the relative motion of the system is

$$T_1 + T_2 + a \gamma_3 = \text{constant},$$

$$T_1 = 1/2 \{ M_1 (v_1^2 + v_2^2 + v_3^2) + A (\omega_1^2 + \omega_2^2) + C \omega_3^2 \},$$

$$T_2 = 1/2 \left\{ e \int_{\tau} (v_x^2 + v_y^2 + v_z^2) d\tau \right\}.$$

The integral of areas is

$$(A\omega_1 + g_1) \gamma_1 + (A\omega_2 + g_2) \gamma_2 + (C\omega_3 + g_3) \gamma_3 = \text{constant}$$

where

$$g_1 = e \int_{\tau} (y v_z - z v_y) d\tau,$$

$$g_2 = e \int_{\tau} (z v_x - x v_z) d\tau,$$

and

$$g_3 = e \int_{\tau} (x v_y - y v_x) d\tau.$$

Since

$$A = B \text{ and } L_3 = 0, \text{ then } \omega_3 \text{ is a constant}$$

and

$$g = \omega e \int_{\tau} (x^2 + y^2) d\tau = \text{constant}.$$

Now we consider the stability of rotation of the solid and the corresponding steady motion of the liquid in its cavity at the equilibrium point:

$$\omega_1 = \omega_2 = 0, \omega_3 = \omega, \gamma_1 = \gamma_2 = 0, \gamma_3 = 1,$$

$$v_1 = v_2 = v_3 = 0, \quad g_1 = g_2 = 0, \quad g_3 = g.$$

For the perturbed motion we shall substitute

$$\omega_3 = \omega + \xi, \quad g_3 = g + \eta, \quad \gamma_3 = 1 + \zeta.$$

Thus, the first integrals given above can be written in the following form for the perturbed motion:

$$V_1 = M_1 (v_1^2 + v_2^2 + v_3^2) + A (\omega_1^2 + \omega_2^2) + C (\xi^2 + 2\omega\xi) + 2T_2 + 2a\xi,$$

$$V_2 = (A\omega_1 + g_1) \gamma_1 + (A\omega_2 + g_2) \gamma_2 + C \xi + \eta + C(\omega + \xi)\zeta + (g + \eta)\zeta,$$

$$V_3 = \gamma_1^2 + \gamma_2^2 + \zeta^2 + 2\zeta = 0, \quad V_4 = \xi = \text{constant}.$$

Also, we consider the function

$$H = M_1 (v_1^2 + v_2^2 + v_3^2) + A (\omega_1^2 + \omega_2^2) + C (\xi^2 + 2\omega\xi) + \\ + \left[\frac{1}{S} \right] (g_1^2 + g_2^2 + 2g\eta + \eta^2) + 2a\xi,$$

where S is proportioned to the greatest principal moment of inertia of the liquid about O . By Liapunov's inequality

$$g_1^2 + g_2^2 + g_3^2 < 2T_2 S$$

we have that $H_1 \leq V_1 = \text{constant}$.

Thus, as a candidate for a "Liapunov function" we consider

$$V = H_1 + 2\lambda V_2 - (a + C\omega\lambda + g\lambda) V_3 + \\ - 2C(\omega + \lambda)V_4 + \frac{C(C - A)}{A} V_4^2,$$

where λ is a constant. By Sylvester's criterion the quadratic part of V is positive definite if there exists a λ such that

$$(1) (A + S)\lambda^2 + (C\omega + g)\lambda + a < 0.$$

The inequality (1) is possible if the left hand side has two distinct real roots λ_1 and λ_2 ; that is if

$$(2) (C\omega + g)^2 - 4(A + S)a > 0.$$

The function V is positive definite in all its variables if (2) is satisfied and if

$$(3) (g/S + \lambda)\eta \geq 0,$$

where $\lambda_1 < \lambda < \lambda_2$. By the theorem stated in the theory part

of this section, and by the strength of the following inequality

$$V \leq V_1 + 2 \lambda V_2 - (a + C \omega \lambda + g \lambda) V_3 + \\ - 2C (\omega + \lambda) V_4 + \frac{C(C - A)}{A} V_4^2,$$

we conclude that the unperturbed motion of the system is stable with respect to

$$\omega_1, \omega_2, \omega_3, g_1, g_2, g_3, \gamma_1, \gamma_2, \gamma_3, v_1, v_2, v_3.$$

Example 24, [22, 23, 24]

Magnetohydrodynamics

In reference [22], Bernstein and others proved an energy principle for magnetohydrodynamics based upon the series expansion in terms of small displacements, $\xi(\underline{r}, t)$, of an ideal conducting fluid along a complete system of normal vibrations. In [23] and [24], Stepanov and Khomeniuk produce proofs of the stability theorems of the equilibrium configurations of an ideal conducting fluid by using Liapunov functions.

In [22], Bernstein started his derivation from basic fluid motion equations and Maxwell's equations in Electromagnetic field theory. The fluid was described by the Lagrangian description; thus, all quantities in the above basic equations become functions of \underline{r}_0 . For the small displacement analysis, the displacement \underline{r} of any fluid particle, \underline{r}_0 , at any time t was expressed as

$$\underline{r} = \underline{r}_0 + \underline{\xi}(\underline{r}_0, t), \quad \underline{\xi}(\underline{r}_0, 0) = \underline{0}.$$

The resulting equations of motion for the small displacements were finally reduced to

$$\rho_0 \ddot{\underline{\xi}} = \underline{F}(\underline{\xi}),$$

where \underline{F} must be a self-adjoint operator. That is, for any two vector fields $\underline{\xi}$ and $\underline{\eta}$ the following equation holds, the integration being over the entire volume in question,

$$\int \underline{\eta} \cdot \underline{F}(\underline{\xi}) d\tau = \int \underline{\xi} \cdot \underline{F}(\underline{\eta}) d\tau.$$

The problem of Bernstein was based on the fact that it is possible in principle to follow in time any small motion about an equilibrium state in which the fluid velocity is zero. The central problem is then to determine for a given equilibrium configuration whether such a small motion grows in time. If the details of the fluid motion are not needed, then all one has to do for stability studies is to examine the sign of the change in the potential energy, which is a functional of ξ . The theorem considered in [22] says, "the system is unstable if and only if there exists some displacement ξ which makes the change in potential energy negative."

In reference [23], Stepanov and Khomeniuk, by the direct method of Liapunov, show that an equilibrium state of an ideally conducting fluid is unstable if there exists displacements of the fluid $\xi(r)$ from the equilibrium position for which the potential energy of the system decreases ($U(\xi) < 0$). These authors use the same equations of motion as Bernstein:

$$\rho \ddot{\xi} = F(\xi),$$

where F is a linear self-adjoint operator. F is defined by

$$\begin{aligned} F(\xi) = & \nabla(\xi \cdot \nabla p) + \gamma \nabla(p \nabla \cdot \xi) + \\ & + \left[\frac{1}{4\pi} \right] (\text{rot rot}(\xi \times H) \times H) + \left[\frac{1}{4\pi} \text{rot } H \right] \times \\ & \times [\text{rot}(\xi \times H)], \end{aligned}$$

where ρ , p , H are equilibrium values of density, pressures of fluid and magnetic fields, and γ is the adiabatic exponent.

We assume that the fluid occupies a finite volume V and is bounded by surface S ; the density ρ and displacement ξ are zero on S .

The equation $\rho \ddot{\xi} = F(\xi)$ has an energy integral

$$E = T + U = \text{constant},$$

where

$$T(\xi) = \frac{1}{2} \int_V \rho \|\dot{\xi}\|^2 dr, \quad U(\xi) = -1/2 \int_V \|\xi\| \cdot \|F(\xi)\| dr$$

It is clear that

$$T = U = 0 \text{ for } \xi = \dot{\xi} = 0,$$

$$T \rightarrow 0, \quad U \rightarrow 0 \text{ for } \|\xi\| \rightarrow 0 \text{ and } \|\dot{\xi}\| \rightarrow 0,$$

$$T(\xi) \geq 0.$$

For any \underline{r} in V let $\underline{\xi}(r)$ be twice continuously differentiable and let

$$\underline{\xi} = \underline{\xi}(t, \underline{r}, \underline{\xi}_0(r), \dot{\underline{\xi}}_0(r))$$

be a solution of $\rho \ddot{\underline{\xi}} = F(\underline{\xi})$ satisfying

$$\underline{\xi} = \underline{\xi}_0(r), \quad \dot{\underline{\xi}} = \dot{\underline{\xi}}_0(r) \text{ for } t = 0.$$

Definition

The equilibrium solution is stable ($\xi = \dot{\xi} = 0$) if for any

$\epsilon_1, \epsilon_2 > 0$ may be found $\delta_1, \delta_2 > 0$ such that if

$$\|\underline{\xi}_0(r)\| < \delta_1, \text{ and } \|\dot{\underline{\xi}}_0(r)\| < \delta_2, \text{ then}$$

$$\|\underline{\xi}(t, \underline{r}, \underline{\xi}_0(r), \dot{\underline{\xi}}_0(r))\| < \epsilon_1, \quad \|\dot{\underline{\xi}}(t, \underline{r}, \underline{\xi}_0(r), \dot{\underline{\xi}}_0(r))\| < \epsilon_2$$

for $t \geq 0$.

Definition

The condition of equilibrium is not stable if there exists at least

one set of $\epsilon_1, \epsilon_2 > 0$ such that for any $\delta_1, \delta_2 > 0$ there

always exist some $\underline{\xi}_0(r)$ and $\dot{\underline{\xi}}_0(r)$, $\|\underline{\xi}_0(r)\| < \delta_1$, and

$\|\dot{\underline{\xi}}_0(r)\| < \delta_2$, such that at least one of the following inequalities

hold:

$$\|\underline{\xi}(t, \underline{r}, \underline{\xi}_0(r), \dot{\underline{\xi}}_0(r))\| \geq \epsilon_1,$$

$$\|\dot{\underline{\xi}}(t, \underline{r}, \underline{\xi}_0(r), \dot{\underline{\xi}}_0(r))\| \geq \epsilon_2,$$

for at least one value of $t \geq 0$.

Theorem 1

If there exists a $\xi(r)$ such that $U(\xi) < 0$, then the equilibrium solution ($\xi = \dot{\xi} = 0$) is not stable.

Proof (Outline)

Let the V function be defined by

$$V(\xi, \dot{\xi}) = \int_V \rho \xi \dot{\xi} dr,$$

and the time derivative is given by

$$\dot{V} = 2 \left\{ T(\dot{\xi}) - U(\xi) \right\}.$$

By hypothesis, \dot{V} and V can have the same sign in the neighborhood of $\xi = \dot{\xi} = 0$. The theorems on instability then conclude that the system is unstable.

The next theorem investigates the influence of viscosity forces on the stability of the equilibrium solution. The equations of motion in this case are

$$\rho \ddot{\xi} = F(\xi) + f(\dot{\xi}),$$

where the force of viscous friction f_i equals

$$f_i = \left\{ \frac{\partial}{\partial x_k} \left[\eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x_l} \right) + \frac{\partial}{\partial x_i} \left[\zeta \frac{\partial v_l}{\partial x_l} \right] \right] \right\};$$

$$i, k, l = 1, 2, 3, \text{ (sum on } k, l).$$

The first and second coefficients of viscosity are η and ζ , and

$$v_i = \dot{\xi}_i. \quad \text{From the above equation of motion it is very easily}$$

found that

$$\frac{d}{dt} [T + U] = -W \leq 0,$$

where

$$W(\dot{\xi}) = - \int_V v_i f_i dr = \\ = \frac{1}{2} \int_V \eta \left[\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x_l} \right]^2 dr + \int_V \zeta \left[\frac{\partial v_l}{\partial x_l} \right]^2 dr.$$

Theorem 2

If there exists $\xi(r)$ such that $U \{ \xi(r) \} < 0$, then the equilibrium solution is not stable even in the presence of viscous forces.

Proof (Outline)

We assume that the condition of equilibrium is stable. Choose $\epsilon_1, \epsilon_2 > 0$; then there exists $\delta_1, \delta_2 > 0$ such that if $\| \xi_0(r) \| < \delta_1$, $\| \dot{\xi}_0(r) \| < \delta_2$, then for $t \geq 0$, $\| \xi(t, r, \xi_0, \dot{\xi}_0) \| < \epsilon_1$, $\| \dot{\xi}(t, r, \xi_0, \dot{\xi}_0) \| < \epsilon_2$.

Assume the V - function takes the form:

$$V \{ \xi, \dot{\xi} \} = \int_V \rho \| \xi \|^2 \cdot \| \dot{\xi} \|^2 dr + \frac{1}{4} \int_V \eta \left[\frac{\partial \xi_i}{\partial x_k} + \frac{\partial \xi_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial \xi_l}{\partial x_l} \right]^2 dr \\ + \frac{1}{2} \int_V \zeta \left[\frac{\partial \xi_l}{\partial x_l} \right]^2 dr.$$

If ξ satisfies the motion equations, then $\dot{V} = 2(T - U)$. There exists ξ_0^* and $\dot{\xi}_0^*$ such that $(\mu = -T - U > 0)$ and $(V > 0)$.

Since for $t \geq 0$,

$$\| \xi(t, r, \xi_0^*, \dot{\xi}_0^*) \| < \epsilon_1, \quad \| \dot{\xi}(t, r, \xi_0^*, \dot{\xi}_0^*) \| < \epsilon_2;$$

then there exists a $\lambda > 0$ such that for $t \geq 0$

$$V \left\{ \underline{\xi}(t, \underline{r}, \underline{\xi}_0^*, \dot{\underline{\xi}}_0^*), \dot{\underline{\xi}}(t, \underline{r}, \underline{\xi}_0^*, \dot{\underline{\xi}}_0^*) \right\} < \lambda,$$

if $\left| \frac{\partial \xi_i}{\partial x_j} \right|$, $i, j = 1, 2, 3$, are smaller than some constant.

On the other hand, since for $t \geq 0$,

$$T \left\{ \dot{\underline{\xi}}(t, \underline{r}, \underline{\xi}_0^*, \dot{\underline{\xi}}_0^*) \right\} + U \left\{ \underline{\xi}(t, \underline{r}, \underline{\xi}_0^*, \dot{\underline{\xi}}_0^*) \right\} \leq -H < 0.$$

We find that for $t \geq 0$,

$$\frac{dV}{dt} \geq 2H > 0 \quad \text{for } \underline{\xi}_0^* \text{ and } \dot{\underline{\xi}}_0^*.$$

Consequently, for $t \longrightarrow \infty$, $V \longrightarrow \infty$. This contradicts the boundedness of V , thus the equilibrium is unstable.

From reference [24] we obtain the theorems dealing with the stability of the equilibrium solution. We are still dealing with ideally conducting fluids. The motion equations are as before:

$$\rho \ddot{\underline{\xi}} = \underline{F}(\underline{\xi}) + \underline{f}(\dot{\underline{\xi}}).$$

The solution of this equation which satisfies the boundary conditions and the initial conditions $\underline{\xi} = \underline{\xi}_0(\underline{r})$, $\dot{\underline{\xi}} = \dot{\underline{\xi}}_0(\underline{r})$, at $t = 0$, is denoted by $\underline{\xi}(t, \underline{r}, \underline{\xi}_0(\underline{r}), \dot{\underline{\xi}}_0(\underline{r}))$. We assume that the solution is twice continuously differentiable with respect to x_k , defined for all $t \geq 0$. W , T , and U are defined as in the previous reference, [23].

We now define two metrics which will be used to define stability in the above system:

$$C_1 \left\{ \underline{\xi}(\underline{r}) \right\} = \int_V \xi^2 dr + \alpha \sum_{k=1}^3 \int_V \left[\frac{\partial \xi}{\partial x_k} \right]^2 dr,$$

$$C_2 \left\{ \dot{\underline{\xi}}(\underline{r}) \right\} = \int_V \rho \dot{\xi}^2 dr,$$

where α is a constant. In the definitions which follow the numbers $\epsilon > 0$ and $\delta > 0$ are always bounded from above by a positive number σ .

Definition

An equilibrium position is stable, if for any $\epsilon_1, \epsilon_2 > 0$ there exists $\delta_1, \delta_2 > 0$, such that if

$$\rho_1 \left\{ \underline{\xi}_0(\underline{r}) \right\} < \delta_1, \rho_2 \left\{ \dot{\underline{\xi}}_0(\underline{r}) \right\} < \delta_2,$$

then for all $t \geq 0$

$$\rho_1 \left\{ \underline{\xi}(t, \underline{r}, \underline{\xi}_0, \dot{\underline{\xi}}_0) \right\} < \epsilon_1, \rho_2 \left\{ \dot{\underline{\xi}}(t, \underline{r}, \underline{\xi}_0, \dot{\underline{\xi}}_0) \right\} < \epsilon_2.$$

Definition

An equilibrium position is unstable, if there exists at least one of the numbers $\epsilon_1, \epsilon_2 > 0$, so that for any $\delta_1 > 0$ and $\delta_2 > 0$, there are always such data

$$\underline{\xi}_0(\underline{r}), \dot{\underline{\xi}}_0(\underline{r}) \cdot \rho_1 \left\{ \underline{\xi}_0 \right\} < \delta_1, \rho_2 \left\{ \dot{\underline{\xi}}_0 \right\} < \delta_2,$$

that at least one of the following inequalities

$$\rho_1 \left\{ \underline{\xi}(t, \underline{r}, \underline{\xi}_0, \dot{\underline{\xi}}_0) \right\} \geq \epsilon_1, \rho_2 \left\{ \dot{\underline{\xi}}(t, \underline{r}, \underline{\xi}_0, \dot{\underline{\xi}}_0) \right\} \geq \epsilon_2,$$

holds for at least one value of $t \geq 0$.

Definition

The functional $V \left\{ \underline{\xi}, \dot{\underline{\xi}} \right\}$ is called positive definite with respect to the metric $\rho \left\{ \underline{\xi}, \dot{\underline{\xi}} \right\}$, if $V \geq 0$ for all admissible $\underline{\xi}$ and $\dot{\underline{\xi}}$; and if for any $\epsilon > 0$, there exists $\lambda(\epsilon) > 0$ such that $V \geq \lambda$ for any $\underline{\xi}, \dot{\underline{\xi}}$ satisfying the condition $\rho \left\{ \underline{\xi}, \dot{\underline{\xi}} \right\} \geq \epsilon$.

An example is the functional $T \left\{ \dot{\underline{\xi}}(\underline{r}) \right\}$, kinetic energy; T is positive definite with respect to the metric $\rho_2 \left\{ \dot{\underline{\xi}} \right\}$.

We now consider the conditions for stability of the equilibrium positions of an ideally conducting inviscid fluid ($\eta = \xi = 0$).

Theorem 3

In order for an equilibrium position of an ideally conducting inviscid fluid to be stable, it is necessary that $U(\underline{\xi}) \geq 0$ for all admissible $\underline{\xi}(r)$.

Proof (Outline)

The proof is a contradiction proof; that is, $U(\underline{\xi})$ is assumed negative for some $\underline{\xi}^*$ which is nonzero. The V - function which is used is

$$V \{ \underline{\xi}, \dot{\underline{\xi}} \} = \int_V \rho \|\underline{\xi}\| \cdot \|\dot{\underline{\xi}}\| dr.$$

The remainder of the proof is similar to theorem 1.

Theorem 4 (Sufficient Condition)

If $U(\underline{\xi})$ is a positive definite functional with respect to the metric

$\rho_1 \{ \underline{\xi} \}$, then the equilibrium position of an ideally conducting inviscid fluid is stable.

Proof (Outline)

This proof is also a contradiction proof. It is shown that for any $\epsilon_1, \epsilon_2 > 0$ and the corresponding $\delta_1 > 0, \delta_2 > 0$, the following inequalities lead to a contradiction and consequently the system is proved to be stable: for some $t = \tau$

$$\rho_1 \{ \underline{\xi}(\tau, \underline{r}, \underline{\xi}_0, \dot{\underline{\xi}}_0) \} \geq \epsilon_1, \rho_2 \{ \dot{\underline{\xi}}(\tau, \underline{r}, \underline{\xi}_0, \dot{\underline{\xi}}_0) \} \geq \epsilon_2,$$

where

$$\rho_1 \{ \underline{\xi}_0 \} < \delta_1, \quad \rho_2 \{ \dot{\underline{\xi}}_0 \} < \delta_2.$$

The vehicle used in arriving at the contradiction is a V - functional defined by

$$V \left\{ \underline{x}, \underline{\dot{x}} \right\} = T \left\{ \underline{\dot{x}} \right\} + U \left\{ \underline{x} \right\},$$

where T and U are positive definite functionals with respect to the metrics

$$\rho_1 \left\{ \underline{\dot{x}} \right\} \text{ and } \rho_2 \left\{ \underline{\dot{x}} \right\}, \text{ respectively.}$$

Theorem 5

If $U(\underline{x})$ is a positive definite functional with respect to the metric $\rho_1 \left\{ \underline{x} \right\}$, then the equilibrium position is stable with viscosity present.

Proof

The proof is similar to that of theorem 4 and the same V - function is used, as well.

Example 25, [57] Van der Pol's Equation

We consider the equation

$$\ddot{x} + \epsilon(1 - x^2) \dot{x} + x = 0,$$

or in state variable notation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - \epsilon(1 - x_1^2) x_2.$$

Since this system does not possess a time-independent integral, a "nearby" system is considered, namely

$$\dot{x}_1 = x_2 + \epsilon \left(x_1 - \frac{x_1^3}{3} \right) + f_4(x_1, x_2)$$

$$\dot{x}_2 = -x_1 - \epsilon(1 - x_1^2) x_2 + f_5(x_1, x_2).$$

This "nearby" system possesses a time-independent integral if

$$\frac{\partial f_4(x_1, x_2)}{\partial x_1} + \frac{\partial f_5(x_1, x_2)}{\partial x_2} = 0.$$

To guarantee that the new system is "nearby", we must determine f_4 and f_5 such that the cross product of the vectors $(\dot{x}_1, \dot{x}_2, 0)$ from the old and new systems is positive semidefinite. This cross product (third component) is

$$X^* = \epsilon x_1^2 (1 - x_1^2/3) + \epsilon^2 x_1 x_2 (1 - x_1^2/3)(1 - x_1^2/3) + x_2 f_5(x_1, x_2) + x_1 f_4(x_1, x_2) + \epsilon(1 - x_1^2) x_2 f_4(x_1, x_2).$$

In the neighborhood of the origin, X^* is positive semidefinite and

$$\left[\frac{\partial f_4}{\partial x_1} + \frac{\partial f_5}{\partial x_2} = 0 \right] \quad \text{is satisfied if}$$

$$f_4(x_1, x_2) = 0,$$

$$f_5(x_1, x_2) = -\epsilon^2 x_1 \left(1 - \frac{x_1^2}{3}\right) \left(1 - x_1^2\right);$$

and since

$$X^* = \epsilon x_1^2 \left(1 - \frac{x_1^2}{3}\right),$$

then

$$0 \leq x_1^2 \leq 3.$$

Therefore, the nearby system becomes

$$\dot{x}_1 = x_2 + \epsilon \left(x_1 - \frac{x_1^3}{3}\right)$$

$$\dot{x}_2 = -x_1 - \epsilon(1 - x_1^2) x_2 - \epsilon^2 \left(x_1 - \frac{x_1^3}{3}\right) (1 - x_1^2),$$

and its first integral is

$$h(x_1, x_2) = x_1^2 + \left[x_2 + \epsilon \left(x_1 - \frac{x_1^3}{3}\right)\right]^2.$$

Let h be the Liapunov function for the original system, where for this system

$$\dot{h} = -2\epsilon x_1^2 \left(1 - \frac{x_1^2}{3}\right).$$

Thus, the original system is asymptotically stable if $\epsilon > 0$ and $|x_1| < \sqrt{3}$.

The region of asymptotic stability predicted by this analysis has the closed boundary defined by

$$x_1^2 + \left[x_2 + \epsilon \left(x_1 - \frac{x_1^3}{3}\right)\right]^2 = 3$$

for any given positive ϵ .

Example 26, [57] A Symmetrical Oscillator

The describing equation of the system is

$$\ddot{x} + d\dot{x} + x - x^3 = 0,$$

or in state variable notation

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_1^3 - dx_2.\end{aligned}$$

This system is symmetrical about the origin in x_1x_2 - space. In this case the nearby system is

$$\begin{aligned}\dot{x}_1 &= x_2 + dx_1 + f_4(x_1, x_2) \\ \dot{x}_2 &= -x_1 + x_1^3 - dx_2 + f_5(x_1, x_2),\end{aligned}$$

where $\frac{\partial f_4}{\partial x_1} + \frac{\partial f_5}{\partial x_2} = 0$. The third component of the cross-

product (as defined above) becomes

$$\begin{aligned}X^* &= dx_1^2 (1 - x_1^2) + d^2 x_1 x_2 + x_1 (1 - x_1^2) f_4(x_1, x_2) + \\ &+ dx_2 f_4(x_1, x_2) + x_2 f_5(x_1, x_2).\end{aligned}$$

The positive semidefiniteness of X^* and the conditions on f_4 and f_5 are satisfied if

$$f_4(x_1, x_2) = 0, \quad f_5(x_1, x_2) = -d^2 x_1^2.$$

Thus, the nearby system becomes

$$\begin{aligned}\dot{x}_1 &= x_2 + dx_1 \\ \dot{x}_2 &= -x_1 + x_1^3 - dx_2 - d^2 x_1,\end{aligned}$$

whose first integral can be found to be

$$h_1(x_1, x_2) = x_1^2 \left[(1 + d^2) - \frac{x_1^2}{2} \right] + 2d x_1 x_2 + x_2^2.$$

We then consider h_1 to be a candidate for a Liapunov function of the original system, where \dot{h}_1 becomes

$$\dot{h}_1 = -2d x_1^2 (1 - x_1^2).$$

Therefore, the original system is asymptotically stable if $d > 0$ and $x_1^2 < 1$. The boundary of the domain of asymptotic stability given by this method is

$$x_1^2 \left[(1 + d^2) - \frac{x_1^2}{2} \right] + 2d x_1 x_2 + x_2^2 = \left[\frac{1 + 2d^2}{2} \right].$$

We could also choose the unknown functions to be

$$f_4(x_1, x_2) = 0$$

$$f_5(x_1, x_2) = -d^2 x_1^2 - dx_1^3,$$

giving

$$X^* = dx_1^2 (1 - x_1^2 - x_1 x_2).$$

The integral corresponding to this choice of f_4 and f_5 for the nearby system is

$$h_2(x_1, x_2) = x_1^2 \left[(1 + d^2) - \frac{x_1^2}{2} (1 - d) \right] + 2dx_1x_2 + x_2^2.$$

The cross product is positive semidefinite if $d > 0$ and $|x_1| < |x_2|$; thus, the resulting boundary of the domain of asymptotic stability of the original system is

$$x_1^2 \left[(1 + d^2) - \frac{x_1^2}{2} (1 - d) \right] + 2dx_1x_2 + x_2^2 < \left[\frac{2d^2 + d + 1}{2} \right].$$

In conclusion, we can take the set-theoretic union of these two domains of asymptotic stability and use this union as a better approximation for the actual domain of asymptotic stability of the system.

Example 27, [57] A Nonsymmetrical Oscillator

The system's equation is

$$\ddot{x} + a\dot{x} + bx + x^2 = 0, \quad a > 0, b > 0,$$

or in state variable notation,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -bx_1 - x_1^2 - ax_2. \end{aligned}$$

The nearby system is defined by

$$\begin{aligned} \dot{x}_1 &= x_2 + ax_1 + f_4(x_1, x_2), \\ \dot{x}_2 &= -bx_1 - x_1^2 - ax_2 + f_5(x_1, x_2), \end{aligned}$$

where $\left[\frac{\partial f_4}{\partial x_1} + \frac{\partial f_5}{\partial x_2} \right] = 0$. As in the previous examples

we choose f_4 and f_5 to satisfy certain conditions; thus, we have

$$f_4(x_1, x_2) = 0, \quad f_5(x_1, x_2) = -a^2 x_1,$$

where

$$\dot{x}^* = ax_1^2 (b + x_1).$$

Thus, the integral of the nearby system and the Liapunov function of the original system is

$$h(x_1, x_2) = \left[\frac{b+a^2}{2} x_1^2 + \frac{x_1^3}{3} + ax_1x_2 + \frac{x_2^2}{2} \right],$$

where

$$\dot{h} = -ax_1^2 (b + x_1) \text{ and } x_1 > -b.$$

The boundary of the domain of asymptotic stability is given by

$$(ax_1 + x_2)^2 + x_1^2 (b + \frac{2}{3}x_1) = b^2 (a^2 + \frac{b}{3}),$$

$$x_1 = -b, \quad 0 \leq x_2 \leq 2ab.$$

It is noted that if $a = 0$ we have a domain of stability and not asymptotic stability.

Example 28, [57] A Nonlinear Compensator

The state space description of this system is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + x_1^3 + dx_2 - \text{sgn}(x_1 + x_2)$$

where $d > 0$. The nearby system is given as

$$\dot{x}_1 = x_2 - dx_1 + \text{sgn}(x_1 + x_2) + f_4(x_1, x_2),$$

$$\dot{x}_2 = x_1 + x_1^3 - \text{sgn}(x_1 + x_2) + f_5(x_1, x_2).$$

From the cross product term we choose

$$f_4(x_1, x_2) = x_2,$$

$$f_5(x_1, x_2) = -(d^2 + 1)x_1 + x_1^3,$$

which gives

$$\begin{aligned} x^* = 1 - d(x_1^2 + x_2^2) + (1 - d)(x_1 + x_2) \operatorname{sgn}(x_1 + x_2) + \\ + dx_1^4 - x_1^3 \operatorname{sgn}(x_1 + x_2). \end{aligned}$$

The integral of the nearby system and the Liapunov function for the original system is

$$h(x_1, x_2) = \left\{ \frac{2+d}{2} x_1^2 - \frac{x_1^4}{2} - dx_1 x_2 + (x_1 + x_2) \operatorname{sgn}(x_1 + x_2) + x_2^2 \right\}$$

For $d = 1$, the domain of asymptotic stability can be shown to be (as given by this analysis)

$$\begin{aligned} \left(\frac{1}{2} \right) (x_1^2 + x_2^2) + \left(\frac{1}{2} \right) (x_1^2 - x_2^2) + (x_1 + x_2) \operatorname{sgn}(x_1 + x_2) + \\ + \frac{x_1^2}{2} (1 - x_1^2) < 1.91. \end{aligned}$$

Example 29, [58] Lewis Servomechanism

This system is a positioning servomechanism with a nonlinear feedback and is described by

$$\ddot{x} + [2a - b|x|] \dot{x} + x = 0,$$

or in state space

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -[2a - b|x_1|] x_2 - x_1.$$

The nearby system is given by

$$\dot{x}_1 = x_2 + \left[2ax_1 - b \int_0^{x_1} |u| du \right] + f_4(x_1, x_2),$$

$$\dot{x}_2 = -[2a - b|x_1|] x_2 - x_1 + f_5(x_1, x_2).$$

Thus, the cross product is

$$\begin{aligned} \dot{x}^* &= \left[2ax_1 - \frac{bx_1^2}{2} \operatorname{sgn}(x_1) \right] \left[2a - b |x_1| \right] x_2 + \\ &+ x_1 \left[2ax_1 - \frac{bx_1^2}{2} \operatorname{sgn}(x_1) \right] + \\ &+ \left[2a - b |x_1| \right] x_2 f_4(x_1, x_2) + x_1 f_4(x_1, x_2) + x_2 f_5(x_1, x_2), \end{aligned}$$

which indicates that we can let

$$\begin{aligned} f_4 &= 0, \\ f_5 &= - \left[2ax_1 - \frac{bx_1^2}{2} \operatorname{sgn}(x_1) \right] \left[2a - b |x_1| \right], \end{aligned}$$

giving

$$\dot{x}^* = x_1^2 \left[2a - \frac{bx_1}{2} \operatorname{sgn}(x_1) \right].$$

Therefore, the choice for the Liapunov function for the original system is

$$h(x_1, x_2) = x_1^2 + \left[x_2 + (2ax_1 - b/2 \operatorname{sgn}(x_1) x_1^2) \right]^2,$$

where

$$2a - \left[\frac{bx_1}{2} \right] \operatorname{sgn}(x_1) > 0.$$

From $h(x_1, x_2)$ and the accompanying condition on x_1 , we define a domain of asymptotic stability by the inequality

$$x_1^2 + \left[x_2 + (2ax_1 - b/2 \operatorname{sgn}(x_1) x_1^2) \right]^2 < \left[\frac{6a^2}{b^2} \right]$$

Example 30, [58] A Nonlinear Damped Pendulum

The equation of this pendulum is

$$\ddot{x} + (\epsilon \sin x) \dot{x} + \sin x = 0,$$

where

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -\sin x_1 - \epsilon x_2 \sin x_1.$$

The cross product becomes

$$\dot{x}^* = \epsilon \sin^2 x_1 + \epsilon^2 x_2 \sin x_1 \cos x_1 + x_2 f_5(x_1, x_2) + [\sin x_1 + \epsilon x_2 \cos x_1] f_4(x_1, x_2).$$

A reasonable choice for the unknown functions is

$$f_4 = 0, f_5 = -\epsilon^2 \sin x_1 \cos x_1,$$

giving

$$\dot{x}^* = \epsilon \sin^2 x_1.$$

The corresponding first integral gives the following candidate for a Liapunov function for the pendulum:

$$h(x_1, x_2) = \frac{x_2^2}{2} + \epsilon x_2 \sin x_1 - \cos x_1 - \frac{\epsilon^2}{4} \cos 2x_1 + 1 + \epsilon^2/4 = \frac{1}{2} (x_2 + \epsilon \sin x_1)^2 + (1 - \cos x_1),$$

where $\epsilon > 0$. The domain of asymptotic stability is given by

$$|x_1| < \pi, \epsilon > 0,$$

$$\frac{x_2^2}{2} + \epsilon x_2 \sin x_1 - \cos x_1 - \epsilon^2/4 \cos 2x_1 < 1 - \epsilon^2/4.$$

Example 31, [58] Globally Stable Oscillator

The describing equations are

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \epsilon(1 - x_1^2 + x_1^4) x_2 - x_1^3. \end{aligned}$$

The nearby system becomes

$$\begin{aligned} \dot{x}_1 &= x_2 - \epsilon(x_1 - x_1^3/3 + x_1^5/5) + f_4(x_1, x_2), \\ \dot{x}_2 &= \epsilon(1 - x_1^2 + x_1^4) x_2 - x_1^3 + f_5(x_1, x_2). \end{aligned}$$

From these systems the cross product term is

$$\begin{aligned} \dot{X}^* = & \epsilon^2 x_1 x_2 (1 - x_1^2/3 + x_1^4/5) (1 - x_1^2 + x_1^4) + \\ & - \epsilon x_1^4 (1 - x_1^2/3 + x_1^4/5) + x_2 f_5(x_1, x_2) + \\ & + \left[x_1^3 \epsilon (1 - x_1^2 + x_1^4) \right] f_4(x_1, x_2). \end{aligned}$$

By inspection of \dot{X}^* , the $x_1 x_2$ - term is eliminated if

$$f_4(x_1, x_2) = 0$$

$$f_5(x_1, x_2) = - \epsilon^2 x_1 (1 - x_1^2/3 + x_1^4/5) (1 - x_1^2 + x_1^4).$$

Thus,

$$\dot{X}^* = - \epsilon x_1^4 \left[\left(\frac{x_1^2}{\sqrt{5}} - \frac{\sqrt{5}}{6} \right)^2 + \frac{31}{36} \right]$$

which implies that $\epsilon < 0$ for a positive semidefinite X^* . The Liapunov function is taken as

$$h(x_1, x_2) = \left[x_1 - \left(x_1^3 - x_1^3/3 + x_1^5/5 \right) \right]^2 + \frac{x_1^4}{2} \geq 0.$$

Therefore, our system is globally asymptotically stable if $\epsilon < 0$.

Example 32, [59] Liquid Motion in a Surge-Tank

The differential equation describing the motion of the water level of a simple surge-tank is highly nonlinear. The results of the authors', [59], analysis show the existence of three positions of equilibrium, and possibly the presence of a limit cycle. In the following, we give the nondimensional quantities involved in the analysis:

τ = nondimensional time,

x = nondimensional water surface level in the tank,

β = nondimensional head-loss in the system,

α = nondimensional velocity.

The equation of motion is given as

$$\ddot{x} + \frac{\alpha^2}{\beta} \dot{x}^2 + \frac{\beta}{\alpha} \frac{\dot{x}}{(1+x)^2} \left[\frac{2\alpha^2}{\beta} - 1 + \frac{2\alpha^2}{\beta} x \right] + \left[x - \beta + \frac{\beta}{(1+x)^2} \right] = 0.$$

In state variable notation we have

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - \left[x_1 - \beta + \frac{\beta}{(1+x_1)^2} \right] - \frac{\alpha^2}{\beta} x_2^2 + \frac{\beta}{\alpha} \frac{x_2}{(1+x_1)^2} \left[\frac{2\alpha^2}{\beta} - 1 + \frac{2\alpha^2}{\beta} x_1 \right].$$

This system describes the motion of the nondimensional water level in the surge-tank, x , and its corresponding velocity, $\dot{x} = \frac{dx}{d\tau}$.

From the above equations it follows that the system has three equilibrium solutions, or singular points. In the $x_1 x_2$ - space the coordinates of the singular points are

$$P_1 = (0, 0), \quad P_2 = \left(-1 + \frac{\beta}{2} \left[1 + (1 + 4/\beta)^{\frac{1}{2}} \right], 0 \right)$$

$$P_3 = \left(-1 + \beta/2 \left[1 - (1 + 4/\beta)^{\frac{1}{2}} \right], 0 \right).$$

The linearized system about point P_1 is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \beta/\alpha \left[\frac{2\alpha^2}{\beta} - 1 \right] x_2,\end{aligned}$$

which indicates the following:

$$(1) \quad \frac{2\alpha^2}{\beta} > 1 \quad , \quad P_1 \text{ is a stable node or focus,}$$

$$(2) \quad \frac{2\alpha^2}{\beta} = 1 \quad , \quad P_1 \text{ is a stable center,}$$

$$(3) \quad \frac{2\alpha^2}{\beta} < 1 \quad , \quad P_1 \text{ is unstable.}$$

Transforming P_2 and P_3 , respectively, to the origins of new coordinates and then linearizing the resulting systems produce the following:

$$(4) \quad \text{for } 0 < \beta < 1/2 \quad , \quad P_2 \text{ is a saddle point,}$$

$$(5) \quad \text{for } 0 < \beta < 1/2 \quad , \quad P_3 \text{ is either an unstable node or focus.}$$

The equilibrium position P_1 , the origin, is the point of interest since it represents the steady state operation of the surge-tank. The saddle point, P_2 , represents an unstable equilibrium above which the water level rises and below which the level falls. Lastly, P_3 is only of mathematical interest. Thus, the study of stability is concerned with point P_1 .

The authors used the integral technique in obtaining the following Liapunov functions; this technique is written up in the text of this section and occurs in [57, 58]. One Liapunov function which was derived is

$$V_1 = \frac{x_2^2}{2} \exp \left[\frac{2\alpha^2}{\beta} x_1 \right] + \int_0^{x_1} \left[x_1 - \beta + \frac{\beta}{(1+x_1)^2} \right] \exp \left[\frac{2\alpha^2}{\beta} x_1 \right] dx_1,$$

where

$$\dot{V}_1 = -\frac{\beta}{\alpha} \frac{x_2^2}{(1+x_1)^2} \left[\frac{2\alpha^2}{\beta} - 1 + \frac{2\alpha^2}{\beta} x_1 \right] \exp \left[\frac{2\alpha^2}{\beta} x_1 \right].$$

Thus V_1 is positive definite and \dot{V}_1 is negative semidefinite for

$$-1 < \frac{\beta}{2\alpha^2} < x_1.$$

Therefore, a domain of stability is determined by the closed bounding curve given below:

$$V_1 = \int_0^{-1 + \beta/2\alpha^2} \left[x_1 - \beta + \frac{\beta}{(1+x_1)^2} \right] \exp \left[\frac{2\alpha^2}{\beta} x_1 \right] dx_1.$$

We observe that if $\beta < 2\alpha^2$, then a domain of asymptotic stability exists. As $\beta \rightarrow 2\alpha^2$, the implication is that a limit cycle is formed, and, which eventually shrinks to the origin as $\beta = 2\alpha^2$.

A second Liapunov function, which will give a larger domain of asymptotic stability, is

$$V_2 = \frac{x_2^2}{2} \left(1 - \frac{x_2}{3} \right) \exp \left[\frac{2\alpha^2}{\beta} x_1 \right] + \int_0^{x_1} \left[x_1 - \beta + \frac{\beta}{(1+x_1)^2} \right] \exp \left[\frac{2\alpha^2}{\beta} x_1 \right] dx_1$$

where

$$\dot{V}_2 = -x_2^2 \left\{ -\frac{1}{3} \frac{\alpha^2}{\beta} x_2^2 + \frac{\beta}{\alpha} \frac{(1-x_2)}{(1+x_1)^2} \left(\frac{2\alpha^2}{\beta} - 1 + \frac{2\alpha^2}{\beta} x_1 \right) - \left[x_1 - \beta + \frac{\beta}{(1+x_1)^2} \right] \right\}.$$

Example 33, [60] Barbasin's Equation

Barbasin's equation [62] is given as

$$\ddot{x} + a_1 \dot{x} + \phi(\dot{x}) + f(x) = 0,$$

where the zeros of ϕ and f are only at $\dot{x} = 0$ and $x = 0$, respectively, and a_1 is a constant. The conditions derived by Barbasin for global asymptotic stability are herein derived by Walker's and Clark's method.

In state variable notation Barbasin's equation becomes

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3,$$

$$\dot{x}_3 = -ax_3 - \phi(x_2) - f(x_1).$$

We now consider a "nearby" system which possesses a first integral given by the function $H(x_1, x_2, x_3)$. Assuming $\frac{df(x_1)}{dx_1}$ is continuous, the

differential equations defining H are

$$\frac{\partial H}{\partial x_2} = a_1 x_3 + \phi(x_2) + f(x_1)$$

$$\frac{\partial H}{\partial x_1} = \frac{\partial}{\partial x_1} \left[\int \frac{\partial H}{\partial x_2} dx_2 \right] = x_2 \frac{df(x_1)}{dx_1}$$

$$\frac{\partial H}{\partial x_3} = x_3 + \frac{\partial}{\partial x_3} \left[\int \frac{\partial H}{\partial x_2} dx_2 \right] = x_3 + a_1 x_2.$$

Next, we consider as a candidate for the Liapunov function an integral of yet another "nearby" system. This new system is such that the first integral V satisfies the following:

$$\frac{\partial V}{\partial x_1} = \frac{\partial H}{\partial x_1} + f_1 = x_2 \frac{df(x_1)}{dx_1} + f_1$$

$$\frac{\partial V}{\partial x_2} = \frac{\partial H}{\partial x_2} + f_2 = a_1 x_3 + \phi(x_2) + f(x_1) + f_2$$

$$\frac{\partial V}{\partial x_3} = \frac{\partial H}{\partial x_3} + f_3 = x_3 + a_1 x_2 + f_3,$$

where f_i are determined such that \dot{V} is negative semidefinite and

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}.$$

The time derivative, \dot{V} , of V , referred to the original system of Barbashin, is given as

$$\begin{aligned} \dot{V} = (\nabla V)_T \dot{\underline{x}} = & -a_1^2 x_2 x_3 - a_1 x_2 f(x_1) + x_2 f_1 + \\ & + x_3 f_2 - x_2^2 \left[a_1 \frac{\phi(x_2)}{x_2} - \frac{df(x_1)}{dx_1} \right] - f_3 \left[a_1 x_3 + \phi(x_2) + f(x_1) \right]. \end{aligned}$$

Cancellation of the indefinite terms in V is fulfilled if

$$f_1 = a_1 f(x_1)$$

$$f_2 = a_1^2 x_2$$

$$f_3 = 0.$$

The resulting \dot{V} becomes

$$\dot{V} = -x_2^2 \left[a_1 \frac{\phi(x_2)}{x_2} - \frac{df(x_1)}{dx_1} \right],$$

which must be negative semidefinite. By line integration we obtain V :

$$\begin{aligned} V = & \int \frac{\partial V}{\partial x_1} dx_1 + \int \frac{\partial V}{\partial x_2} dx_2 + \int \frac{\partial V}{\partial x_3} dx_3 \\ = & a_1 \int_0^{x_1} f(x_1) dx_1 + x_2 f(x_1) + \frac{a_1^2 x_2^2}{2} + \frac{x_3^2}{2} + a_1 x_2 x_3 + \\ & + \int_0^{x_2} \phi(x_2) dx_2 \\ = & a_1 \int_0^{x_1} f(x_1) dx_1 + x_2 f(x_1) + \int_0^{x_2} \phi(x_2) dx_2 + \frac{1}{2} (a_1 x_2 + x_3)^2. \end{aligned}$$

For global asymptotic stability, V is a proper Liapunov function if

$$(1) \quad a_1 \frac{\phi(x_2)}{x_2} - \frac{df(x_1)}{dx_1} > 0, \quad x_2 \neq 0,$$

$$(2) \quad a_1 > 0,$$

$$(3) \quad x_1 f(x_1) > 0, \quad x_1 \neq 0,$$

$$(4) \quad V \rightarrow \infty \text{ for } \|\underline{x}\| \rightarrow \infty,$$

$$(5) \quad \frac{1}{2} (a_1 x_2 + x_3)^2 + \int_0^{x_2} \phi(x_2) dx_2 + a_1 \int_0^{x_1} f(x_1) dx_1 > x_2 f(x_1).$$

Example 34, [60] A Third Order Example

The defining equation for this example is

$$\ddot{x} + b\dot{x} + (x + c\dot{x})^m = 0$$

or in state variable notation,

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3,$$

$$\dot{x}_3 = -bx_3 - (x_1 + cx_2)^m.$$

The H -function is defined by

$$\frac{\partial H}{\partial x_2} = bx_3 + (x_1 + cx_2)^m$$

where

$$\frac{\partial H}{\partial x_1} = \frac{\partial}{\partial x_1} \left[\int \frac{\partial H}{\partial x_2} dx_2 \right] = \frac{1}{c} (x_1 + cx_2)^m,$$

$$\frac{\partial H}{\partial x_3} = x_3 + \frac{\partial}{\partial x_3} \left[\int \frac{\partial H}{\partial x_2} dx_2 \right] = x_3 + bx_2.$$

For the second "nearby" system the integral V is given by

$$\frac{\partial V}{\partial x_1} = \frac{\partial H}{\partial x_1} + f_1 = \frac{1}{c} (x_1 + cx_2)^m + f_1,$$

$$\frac{\partial V}{\partial x_2} = \frac{\partial H}{\partial x_2} + f_2 = bx_3 + (x_1 + cx_2)^m + f_2,$$

$$\frac{\partial V}{\partial x_3} = \frac{\partial H}{\partial x_3} + f_3 = x_3 + bx_2 + f_3.$$

Considering V as a candidate for a Liapunov function implies that \dot{V} is given as follows:

$$\begin{aligned} \dot{V} = (\nabla V)_T \dot{\underline{x}} &= (1/c - b) x_2 (x_1 + cx_2)^m - b^2 x_2 x_3 + x_2 f_1 \\ &+ x_3 f_2 - f_3 [bx_3 + (x_1 + cx_2)^m]. \end{aligned}$$

First, we see that the first term on the right should be eliminated; thus, choose

$$\begin{aligned} f_2 &= [-b + 1/c] x_3 + g_2, \\ f_3 &= [-b + 1/c] x_2 + g_3, \end{aligned}$$

where $\frac{\partial g_2}{\partial x_3} = \frac{\partial g_3}{\partial x_2}$ must be satisfied if $\frac{\partial f_2}{\partial x_3} = \frac{\partial f_3}{\partial x_2}$.

Therefore, \dot{V} becomes

$$\begin{aligned} \dot{V} &= - [b - 1/c] x_3^2 - b/c x_2 x_3 + x_3 g_2 + x_2 f_1 \\ &- g_3 [bx_3 + (x_1 + cx_2)^m]. \end{aligned}$$

\dot{V} will be negative semidefinite if we retain only the first term on the right; thus, let

$$f_1 = 0, \quad g_3 = 0, \quad g_2 = \left[\frac{bx_2}{c} \right].$$

The final form for \dot{V} is

$$\dot{V} = - \frac{x_2^2}{3} (b - 1/c)$$

which is negative semidefinite if $b > 1/c$.

By line integration of the gradient of V ,

$$(\nabla V)_T = \left\{ \frac{1}{c} [x_1 + cx_2]^m, b/c x_2 + \left(\frac{1}{c}\right) x_3 + [x_1 + cx_2]^m, 1/c x_2 + x_3 \right\},$$

we have

$$V = \frac{1}{c(m+1)} [x_1 + cx_2]^{m+1} + \frac{1}{2} \left[x_3 + \frac{x_2^2}{2} \right]^2 + \frac{1}{2c^2} [bc - 1] x_2^2.$$

The sufficient conditions for this third order nonlinear system to be globally asymptotically stable are

- (1) $bc > 1$,
- (2) $c > 0$,
- (3) m is a positive odd integer.

Example 35, [60] Nonlinear Feedback System

This particular nonlinear feedback system is defined by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -3x_1^2 x_3 - 2x_2 - 6x_1 x_2^2 - x_1^3.$$

The H - function is defined by

$$\frac{\partial H}{\partial x_2} = 3x_1^2 x_3 + 2x_2 + 6x_1 x_2^2 + x_1^3,$$

where

where

$$\frac{\partial H}{\partial x_1} = \frac{\partial}{\partial x_1} \left[\int \frac{\partial H}{\partial x_2} dx_2 \right] = 6x_1 x_2 x_3 + 2x_2^3 + 3x_1^2 x_2 ,$$

$$\frac{\partial H}{\partial x_3} = x_3 + \frac{\partial}{\partial x_3} \left[\int \frac{\partial H}{\partial x_2} dx_2 \right] = x_3 + 3x_1^2 x_3 .$$

The corresponding V - function is given by

$$\frac{\partial V}{\partial x_1} = 3x_1^2 x_2 + 2x_2^3 + 6x_1 x_2 x_3 + f_1$$

$$\frac{\partial V}{\partial x_2} = 3x_1^2 x_3 + 2x_2 + 6x_1 x_2^2 + x_1^3 + f_2 ,$$

$$\frac{\partial V}{\partial x_3} = x_3 + 3x_1^2 x_3 + f_3 .$$

Considering the original system the time derivative of V is

$$\begin{aligned} \dot{V} = (\nabla V)_T \dot{x} = & \left[6x_1 x_2^2 - 9x_1^4 x_2 \right] x_3 + \\ & + \left[2x_2^3 - 18x_1^3 x_2^2 - 3x_1^5 \right] x_2 - 3x_1^2 x_2^2 + \\ & + x_2 f_1 + x_3 f_2 - f_3 \left[3x_1^2 x_3 + 2x_2 + 6x_1 x_2^3 + x_1^3 \right] , \end{aligned}$$

Choose the f_i in the following way:

$$f_1 = - 2x_2^3 + 18x_1^3 x_2^2 + 3x_1^5 ,$$

$$f_2 = - 6x_1 x_2^2 + 9x_1^4 x_2 ,$$

$$f_3 = 0 ,$$

which gives a negative semidefinite \dot{V} ,

$$\dot{V} = - 3x_1^2 x_2^2 .$$

The line integral of ∇V gives

$$V = \frac{1}{2} \left[x_3 + 3x_1^2 x_2 \right]^2 + \left[x_2 + \frac{1}{2} x_1^3 \right]^2 + \frac{1}{4} x_1^6 .$$

Looking at V and \dot{V} , we see that all of the conditions of LaSalle's theorem are satisfied in the entire space, thus concluding that the system is globally asymptotically stable.

Example 36, [60] A Nonsymmetrical System

Consider the third order system defined by

$$\ddot{x} + b\ddot{x} + \dot{x} + x + ax^2 = 0,$$

or in state space notation,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -bx_3 - x_2 - x_1 - ax_1^2 ,$$

For the corresponding nearby system the H - function is given as

$$\frac{\partial H}{\partial x_2} = bx_3 + x_2 + x_1 + ax_1^2 ,$$

where

$$\frac{\partial H}{\partial x_1} = x_2 + 2ax_1 x_2 ,$$

$$\frac{\partial H}{\partial x_3} = x_3 + bx_2$$

Thus, the gradient of the V - function becomes

$$\frac{\partial V}{\partial x_1} = x_2 + 2ax_1 x_2 + f_1 ,$$

$$\frac{\partial V}{\partial x_2} = bx_3 + x_2 + x_1 + ax_1^2 + f_2 ,$$

$$\frac{\partial V}{\partial x_3} = x_3 + bx_2 + f_3 .$$

For the original system, the time derivative of V is

$$\begin{aligned} \dot{V} = & - \left[b-1-2ax_1 \right] x_2^2 - b^2 x_2 x_3 - bx_2 \left[x_1 + ax_1^2 \right] + \\ & + x_2 f_1 + x_3 f_2 - f_3 \left[bx_3 + x_1 + x_2 + ax_1^2 \right]. \end{aligned}$$

Choose the f_i such that the second and third terms, on the right, in \dot{V} cancel; that is,

$$f_1 = b(x_1 + ax_1^2),$$

$$f_2 = b^2 x_2,$$

$$f_3 = 0,$$

and

$$\dot{V} = -x_2^2 \left[b - 1 - 2ax_1 \right].$$

\dot{V} is negative semidefinite if $2ax_1 < b - 1$. Integration of the gradient of V , ∇V , gives

$$\begin{aligned} V = & \frac{1}{2} \left[x_3 + bx_2 \right]^2 + 1/2 \left[x_2 + x_1 + ax_1^2 \right]^2 + \\ & + \frac{1}{2} x_1^2 \left[b - 1 - a^2 x_1^2 + 2a(b/3 - 1)x_1 \right]. \end{aligned}$$

Locally, the origin is asymptotically stable if

$$(1) \quad 2ax_1 < b - 1,$$

$$(2) \quad 0 < b - 1.$$

The region of asymptotic stability about $x = 0$ as given by the above Liapunov function is

$$0 \leq V < \min \left[\frac{(b-1)^3 (b+3)}{96a^2}, \frac{b}{6a^2} \right].$$

Example 37, [60] Linear Switching System

The defining equation of this system contains a function which is piecewise linear, namely:

$$\ddot{x} + b_1 \dot{x} + b_2 \dot{x} + b_3 \operatorname{sgn}(x + cx) = 0,$$

where the b_i 's and c are constants and

$$\operatorname{sgn}(y) = \begin{cases} 1, & y > 0 \\ 0, & y = 0 \\ -1, & y < 0 \end{cases}.$$

In state variable notation we have

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -b_1 x_3 - b_2 x_2 - b_3 \operatorname{sgn}(x_1 + cx_2).$$

The H and V functions are defined as in the previous examples, the gradient of V being given as

$$\frac{\partial V}{\partial x_1} = \frac{b_3}{c} \operatorname{sgn}(x_1 + cx_2) + f_1$$

$$\frac{\partial V}{\partial x_2} = b_1 x_3 + b_2 x_2 + b_3 \operatorname{sgn}(x_1 + cx_2) + f_2$$

$$\frac{\partial V}{\partial x_3} = x_3 + b_1 x_2 + f_3.$$

Thus, the time derivative of V is

$$\begin{aligned} \dot{V} = & -b_3 (b_1 - 1/c) x_2 \operatorname{sgn}(x_1 + cx_2) - b_1^2 x_2 x_3 - b_1 b_2 x_2^2 + \\ & + x_2 f_1 + x_3 f_2 - f_3 \left[b_1 x_3 + b_2 x_2 + b_3 \operatorname{sgn}(x_1 + cx_2) \right]. \end{aligned}$$

We choose the f_i such that the first two terms in V disappear; thus, we have

$$f_1 = 0$$

$$f_2 = - \left[b_1 - 1/c \right] x_3 + \frac{b_1 x_2}{c},$$

$$f_3 = - \left[b_1 - 1/c \right] x_2,$$

where $\left[\frac{\partial f_i}{\partial x_j} \right] = \left[\frac{\partial f_i}{\partial x_1} \right]$. Therefore,

$$\dot{V} = - \left[b_1 - 1/c \right] x_3^2 - \left[\frac{b_2}{c} x_2^2 \right],$$

and integration of (VV) gives

$$V = \frac{b_3}{c} \left[x_1 + cx_2 \right] \operatorname{sgn} (x_1 + cx_2) + \frac{b_1}{2c} \left[x_2 + \frac{x_3}{b_1} \right]^2 + \frac{b_2}{2} x_2^2 + \frac{cb_1 - 1}{2cb_1} x_3^2.$$

The system is globally asymptotically stable for

$$(1) \quad b_1, b_2, b_3 > 0,$$

$$(2) \quad c > 0,$$

$$(3) \quad b_1 c > 1.$$

Example 38, [60] Fourth Order System

Consider the system defined by

$$\ddot{\ddot{x}} + 4\ddot{\ddot{x}} + 5\ddot{x} + 2\dot{x} + cx^3 = 0,$$

or in state variable notation

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3,$$

$$\dot{x}_3 = x_4,$$

$$\dot{x}_4 = -4x_4 - 5x_3 - 2x_2 - cx_1^3.$$

The H - function is defined by

$$\frac{\partial H}{\partial x_3} = 4x_4 + 5x_3 + 2x_2 + cx_1^3,$$

$$\frac{\partial H}{\partial x_1} = \frac{\partial}{\partial x_1} \left[\int \frac{\partial H}{\partial x_3} dx_3 \right] = 3cx_1^2 x_3,$$

$$\frac{\partial H}{\partial x_2} = \frac{\partial}{\partial x_2} \left[\int \frac{\partial H}{\partial x_3} dx_3 \right] = 2x_3,$$

$$\frac{\partial H}{\partial x_4} = x_4 + \frac{\partial}{\partial x_4} \left[\int \frac{\partial H}{\partial x_3} dx_3 \right] = x_4 + 4x_3.$$

The gradient of the V - function is given by

$$\frac{\partial V}{\partial x_1} = \frac{\partial H}{\partial x_1} + f_1 = 3cx_1^2 x_3 + f_1,$$

$$\frac{\partial V}{\partial x_2} = \frac{\partial H}{\partial x_2} + f_2 = 2x_3 + f_2$$

$$\frac{\partial V}{\partial x_3} = \frac{\partial H}{\partial x_3} + f_3 = 4x_4 + 5x_3 + 2x_2 + cx_1^3 + f_3,$$

$$\frac{\partial V}{\partial x_4} = \frac{\partial H}{\partial x_4} + f_4 = x_4 + 4x_3 + f_4$$

The time derivative of V is

$$\begin{aligned} \dot{V} = & -16x_3x_4 - 8x_2x_3 - 4cx_3x_1^3 + 3cx_1^2 x_2x_3 + \\ & -18x_3^2 + x_2f_1 + x_3f_2 + x_4f_3 + \\ & -f_4 \left[4x_4 + 5x_3 + 2x_2 + cx_1^3 \right]. \end{aligned}$$

By choosing the f_i as given below, the first three terms in V are cancelled:

$$f_1 = 12cx_1^2 x_2 + b cx_1^3 + f_1^*,$$

$$f_2 = 8x_2 + 4cx_1^3 + bx_4 + 5bx_2 + 4bx_3 + f_2^*,$$

$$f_3 = 16x_3 - bx_3 + 4bx_2 + f_3^*,$$

$$f_4 = bx_2 + f_4^*,$$

where b is a parameter to be determined and f_i^* are new undetermined functions where must satisfy $\frac{\partial f_i^*}{\partial x_j} = \frac{\partial f_j^*}{\partial x_i}$. Thus, \dot{V} becomes

$$\begin{aligned} \dot{V} = & 3cx_1^2 x_2 x_3 - (18 - 4b) x_3^2 + 12 cx_1^2 x_2^2 + \\ & - 2bx_2^2 + x_2^* f_1^* + x_3^* f_2^* + x_4^* f_3^* + \\ & - f_4^* \left[4x_4 + 5x_3 + 2x_2 + cx_1^3 \right]. \end{aligned}$$

This new expression for \dot{V} is further simplified if

$$f_1^* = f_3^* = f_4^* = 0, \quad f_2^* = -b/2 x_2.$$

A convenient choice for b is $b = (192/43)$; then we have

$$\begin{aligned} \dot{V} = & -12 \left[\frac{32}{43} - cx_1^2 \right] \cdot \left[x_2 + 1/8 x_3 \right]^2 + \\ & - \frac{3}{16} cx_1^2 x_3^2, \end{aligned}$$

which is negative semidefinite for $x_1^2 \leq \frac{32}{43c}$. The corresponding V is

$$\begin{aligned} V = & cx_1^3 x_3 + 4cx_1^3 x_2 + \left(\frac{bc}{4} \right) x_1^4 + 2(1 + 2b) x_2 x_3 + \\ & + \frac{1}{4} (16 + 9b) x_2^2 + b x_2 x_4 + 4x_3 x_4 + \\ & + \frac{1}{2} (21 - b) x_3^2 + 1/2 x_4^2. \end{aligned}$$

A conservative estimate of the domain of asymptotic stability about $\underline{x} = \underline{0}$

for this Liapunov function is given by $V_B = 3/10 \quad c \geq V \geq 0$,

where $C > 0$ and $b = \frac{192}{43}$.

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SECTION THREE

LIAPUNOV'S DIRECT METHOD AND

ROUTH'S CANONICAL FORM

Prepared by: R. L. Drake
T. Gifford

LIAPUNOV'S DIRECT METHOD AND ROUTH'S CANONICAL FORM

SUMMARY

In this section the equivalence between Liapunov's Direct Method and the Routh-Hurwitz Criterion for linear systems is established. For a linear system $\dot{\underline{x}} = \underline{A} \underline{x}$, a transformation matrix \underline{Q} is developed which transforms the system matrix \underline{A} into a special matrix \underline{R} called a Routh Canonical Form. That is, by letting $\underline{y} = \underline{Q} \underline{x}$ and substituting into the system equation, we get $\dot{\underline{y}} = \left[\underline{Q} \underline{A} \underline{Q}^{-1} \right] \underline{y}$, where $\underline{R} = \underline{Q} \underline{A} \underline{Q}^{-1}$. The elements of \underline{R} are closely related to the elements of the first column of Routh's arrays. For linear systems the conditions for stability obtained from the \underline{R} matrix by Liapunov's Direct Method are the same as the Routh-Hurwitz Criterion.

This treatment is then extended to nonlinear systems. As a result of the application of the transformation, $\underline{y} = \underline{Q} \underline{x}$, to nonlinear systems, the linear terms are essentially removed from further consideration and only the nonlinear terms remain. This method is discussed in detail and a compendium of nonlinear differential equations analyzed by this method is presented.

INTRODUCTION

In reference [1]*, the Liapunov's Direct Method is shown to yield necessary and sufficient conditions for the stability of solutions of linear, time-invariant differential equations. These conditions must be equivalent to the Routh-Hurwitz conditions, for these are also necessary and sufficient. Several papers have recently dealt with this equivalence. In England, Parks[2,3] gave a direct link between the two methods by proving the Routh-Hurwitz Criterion using Liapunov's Direct Method. In this country, Reiss and Geiss [4] have given a more straight-forward proof than that of Parks. The equivalence between the two methods of analyzing linear systems presented in this section will follow the work of Puri and Weygandt [5]. The reason

* refer to the references at the end of this section.

for following their work is that it can be extended to the stability analysis of nonlinear systems.

The material presented in this section is based on the discussion and examples found in references [5,6,7,8]. The Routh's Canonical Form is applicable to the study of linear systems and a certain class of nonlinear systems. The basic technique is considered first, the discussion being based on references [5,8]. Then the extensions and modifications, as given in references [6,7], are outlined. The final part of this section is a compendium of examples which comes from references [5,6,7]..

In passing, we make note of other applications of the Routh's Canonical Form. In reference [9], Puri and Weygardt calculate quadratic moments of high order linear systems via Routh canonical transformations and Liapunov functions. In reference [10], Puri and Drake analyze the stability of nonlinear, nonautonomous difference equations by using Routh's Canonical Form to generate Liapunov functions.

BASIC SYSTEM IN JORDAN CANONICAL FORM

The system being analysed is described by a differential equation of the form

$$x^{(n)} + a_1 x^{(n-1)} + \dots + a_{n-1} x^{(1)} + a_n x + F = 0, \quad (1)$$

where $x^{(n)} = \frac{d^n x}{dt^n}$ and $F = F(x^{(n-1)}, x^{(n-2)}, \dots, x; t)$

is a known nonlinear function of x and its derivatives. The values a_1, \dots, a_n are real constants. When F is identically zero, we have a linear autonomous system.

System (1) may always be written in the state variable or the Jordan Canonical Form:

$$\dot{\underline{x}} = \underline{A} \underline{x} - \underline{b} F, \quad (2)$$

where

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \begin{matrix} x_1 = x \\ x_2 = \dot{x} \\ \vdots \\ x_n = x^{(n-1)} \end{matrix}, \quad (3)$$

$$\underline{b} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}, \quad (4)$$

$$\dot{\underline{x}} = \frac{d \underline{x}}{dt}, \quad (5)$$

and

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_2 & -a_1 \end{bmatrix}, \quad (6)$$

and where F is a scalar function.

Linear Autonomous System

Consider the linear, autonomous system corresponding to equation (2), namely;

$$\dot{\underline{x}} = \underline{A} \underline{x}. \quad (7)$$

To analyze the stability of this system, we introduce the transformation

$$\underline{y} = \underline{Q} \underline{x}, \quad (8)$$

where \underline{Q} is a real, nonsingular, constant matrix. Substituting equation (8)

into equation (7) gives

$$\dot{\underline{y}} = \underline{Q} \underline{A} \underline{Q}^{-1} \underline{y} = \underline{R} \underline{y}. \quad (9)$$

where R is chosen as a real matrix expressed in the following form:

$$\underline{R} = \begin{bmatrix} -r_1 & \sqrt{r_2} & 0 & . & . & . & 0 & 0 \\ \sqrt{r_2} & 0 & \sqrt{r_3} & . & . & . & 0 & 0 \\ 0 & -\sqrt{r_3} & 0 & . & . & . & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots \\ 0 & 0 & 0 & . & . & . & 0 & \sqrt{r_n} \\ 0 & 0 & 0 & . & . & . & -\sqrt{r_n} & 0 \end{bmatrix} \quad (10)$$

The elements of R are real quantities related to the Hurwitz determinants of equation (7), as was shown in reference [3]. Because of the importance of this report being used for instructional purposes, the above relationships are repeated in the following paragraphs.

Elements of R vs. Elements of A

The matrix A in equation (6) for orders, 1,2,3, ... can be denoted by

$$\begin{aligned} \underline{A}_1 &= -a_1, \\ \underline{A}_2 &= \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \\ \underline{A}_3 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}, \end{aligned}$$

The characteristic polynomial for the i^{th} order matrix is denoted by

$$\Delta_{A,i}(\lambda) = \left| \lambda \underline{I} - \underline{A}_i \right|, \quad i = 1, 2, 3, \dots$$

In expanded form, we have

$$\begin{aligned} \Delta_{A,1}(\lambda) &= a_1 + \lambda, \\ \Delta_{A,2}(\lambda) &= a_2 + \lambda a_1 + \lambda^2, \end{aligned}$$

$$\Delta_{A,n}(\lambda) = a_n + a_{n-1}\lambda + \dots + a_1\lambda^{n-1} + \lambda^n.$$

The recursion relationships for the characteristic equations are

$$\begin{aligned} \Delta_{A,0}(\lambda) &= 1, \\ \Delta_{A,i}(\lambda) &= a_i + \lambda \Delta_{A,i-1}(\lambda), \quad i = 1, \dots, n. \end{aligned} \tag{11}$$

Similarly, matrix R in equation (10) can be written, for orders 1,2,3,..., as:

$$\begin{aligned} \underline{R}_1 &= r_1, \\ \underline{R}_2 &= \begin{bmatrix} -r_1 & \sqrt{r_2} \\ -\sqrt{r_2} & 0 \end{bmatrix}, \\ \underline{R}_3 &= \begin{bmatrix} -r_1 & \sqrt{r_2} & 0 \\ -\sqrt{r_2} & 0 & \sqrt{r_3} \\ 0 & -\sqrt{r_3} & 0 \end{bmatrix}, \end{aligned}$$

The characteristic polynomials corresponding to R are

$$\Delta_{R,i}(\lambda) = \left| \lambda \underline{I} - \underline{R}_i \right|.$$

The recursion relationships for these characteristic equations are

$$\begin{aligned} \Delta_{R,1}(\lambda) &= r_1 + \lambda, \\ \Delta_{R,2}(\lambda) &= r_2 + r_1\lambda + \lambda^2, \\ \Delta_{R,i}(\lambda) &= \lambda \Delta_{R,i-1}(\lambda) + r_i \Delta_{R,i-2}(\lambda), \quad i = 3, 4, \dots, n. \end{aligned} \tag{12}$$

Since \underline{A}_1 and \underline{R}_1 are similar matrices, we have

$$\Delta_{R,i}(\lambda) = \Delta_{A,i}(\lambda) , \quad (13)$$

$$i = 1, \dots, n.$$

In equation (13), we equate coefficients of equal powers of λ to obtain the relationships between the elements of \underline{R}_i and \underline{A}_i .

As an example, consider $i = 6$. The results from equation (13) are

$$\begin{aligned} a_1 &= r_1 \\ a_2 &= r_2 + r_3 + r_4 + r_5 + r_6 \\ a_3 &= r_1 (r_3 + r_4 + r_5 + r_6) \\ a_4 &= r_4 r_6 + r_3 r_5 + r_3 r_6 + r_2 (r_4 + r_5 + r_6) \\ a_5 &= r_1 r_3 (r_5 + r_6) + r_1 r_4 r_6 \\ a_6 &= r_2 r_4 r_6 . \end{aligned}$$

Solving these equations for r_i in terms of a_i gives:

$$\begin{aligned} r_1 &= \Delta_1 = a_1 \\ r_2 &= \Delta_2 / \Delta_1 = b_1 = a_2 = a_3 / a_1 \\ r_3 &= \Delta_3 / \Delta_1 \Delta_2 = b_3 / a_1 = a_3 / a_1 - b_2 / b_1 \\ r_4 &= \Delta_1 \Delta_4 / \Delta_2 \Delta_3 = b_2 / b_1 - b_4 / b_3 \\ r_5 &= \Delta_2 \Delta_5 / \Delta_3 \Delta_4 = b_3 / b_1 - a_6 b_3 / (b_2 b_3 - b_1 b_4) \\ r_6 &= \Delta_3 \Delta_6 / \Delta_4 \Delta_5 = a_6 / r_2 r_4 , \end{aligned}$$

where Δ_i are the Hurwitz subdeterminants and

$$b_1 = a_2 - a_3 / a_1$$

$$b_2 = a_4 - a_5 / a_1$$

$$b_3 = a_3 - a_1 b_2 / b_1$$

$$b_4 = a_5 - a_1 a_6 / b_1$$

In summary, we have the following results for the n^{th} order system:

$$r_1 = \Delta_1 = a_1$$

$$r_2 = \Delta_2 / \Delta_1 = a_2 - a_3 / a_1$$

$$r_3 = \Delta_3 / \Delta_1 \Delta_2 = a_3 / a_1 - (a_4 - a_5 / a_1) / (a_2 - a_3 / a_1) \quad (14)$$

$$r_i = \Delta_{i-3} \Delta_i / \Delta_{i-2} \Delta_{i-1}, \quad i = 4, 5, \dots, n.$$

Liapunov Function for the Linear System

As a candidate for a Liapunov function, choose:

$$V = \underline{y}_T \underline{y} = \sum_{i=1}^n \left[y_i \right]^2 = \underline{x}_T \underline{P} \underline{x}, \quad (15)$$

where, by equation (8),

$$\underline{P} = \underline{Q}_T \underline{Q}. \quad (16)$$

The time derivative of V is given as

$$\dot{V} = \dot{\underline{y}}_T \underline{y} + \underline{y}_T \dot{\underline{y}} = \underline{y}_T \left[\underline{R}_T + \underline{R} \right] \underline{y}. \quad (17)$$

Combining equations (10) and (17) gives:

$$\dot{V} = -2 r_1 y_1^2, \quad (18)$$

which is negative semi-definite provided that $r_1 > 0$. This is the first Routh-Hurwitz stability condition. The other Routh-Hurwitz conditions are determined from the requirement that \underline{Q} is a real matrix.

The following is an algorithm for finding the elements of Q for an n^{th} order system. This algorithm is derived from the equation

$$Q \underline{A} = \underline{R} Q \quad (19)$$

Let the rows of Q be designated $0, 1, 2, \dots, n-1$. The elements in the i^{th} row of Q are the coefficients of λ in the expressions for $\Delta_{R,1}$ multiplied by

$$\left(\prod_{j=2+i}^n r_j \right)^{1/2} \quad \text{except when } i = n-1, \text{ then the multiplier is unity.}$$

As an example, consider $n=4$; the results are:

$$Q = \begin{bmatrix} (r_2 r_3 r_4)^{1/2} & 0 & 0 & 0 \\ r_1 (r_3 r_4)^{1/2} & (r_3 r_4)^{1/2} & 0 & 0 \\ r_2 (r_4)^{1/2} & r_1 (r_4)^{1/2} & (r_4)^{1/2} & 0 \\ r_1 r_3 & r_2 + r_3 & r_1 & 1 \end{bmatrix} = \begin{bmatrix} (r_2 r_3 r_4)^{1/2} & 0 & 0 & 0 \\ 0 & (r_3 r_4)^{1/2} & 0 & 0 \\ 0 & 0 & r_4^{1/2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ r_1 & 1 & 0 & 0 \\ r_2 & r_1 & 1 & 0 \\ r_1 r_3 & r_2 + r_3 & r_1 & 1 \end{bmatrix}$$

Since the Q for any order system can be written in this form and since Q is a real matrix, then we require that

$$r_i > 0, \quad i = 1, 2, \dots, n. \quad (20)$$

The inequalities in (20) are exactly the Routh-Hurwitz conditions (see equation (14)). Therefore the relation between Liapunov's Direct Method and the Routh-Hurwitz criterion for linear, autonomous systems has been demonstrated through the use of Routh's Canonical Form.

Nonlinear, Autonomous Systems

For this case we consider equation (2):

$$\dot{\underline{x}} = \underline{A} \underline{x} - \underline{b} F, \quad (2)$$

where F is an autonomous function. The reasoning behind the following procedure can be best expressed by a quote from reference [5] , "since with F identically zero the transformation $\underline{y} = \underline{Q} \underline{x}$ resulted in both necessary and sufficient conditions for asymptotic stability, it is reasonable to expect that the same transformation might be useful in the nonlinear case". Thus, our candidate for a Liapunov function is

$$V_1 = \underline{y}_T \underline{y} = \underline{x}_T \underline{P} \underline{x}, \quad (21)$$

where $\underline{P} = \underline{Q}_T \underline{Q}$ and \underline{Q} is the matrix obtained for the corresponding linear system.

Taking the time derivative of V_1 gives

$$\dot{V}_1 = \dot{\underline{y}}_T \underline{y} + \underline{y}_T \dot{\underline{y}}. \quad (22)$$

Premultiplying equation (2) by \underline{Q} gives

$$\underline{Q} \dot{\underline{x}} = \underline{Q} \underline{A} \underline{x} - \underline{Q} \underline{b} F. \quad (23)$$

Substituting equation (8) into (23) gives

$$\dot{\underline{y}} = \underline{R} \underline{y} - \underline{Q} \underline{b} F, \quad (24)$$

but we see that

$$\underline{Q} \underline{b} = \underline{b}, \quad (25)$$

thus

$$\dot{\underline{y}} = \underline{R} \underline{y} - \underline{b} F. \quad (26)$$

Now substituting (26) into (22) gives

$$\begin{aligned} \dot{V}_1 &= \underline{y}_T [\underline{R} + \underline{R}_T] \underline{y} - [\underline{b}_T \underline{y} + \underline{y}_T \underline{b}] F \\ &= -2r_1 y_1^2 - 2y_n F. \end{aligned} \quad (27)$$

The variables y_1 and y_n can be expressed in terms of x_1, \dots, x_n by using equation (8) and the Q matrix. Depending upon the nature of the nonlinearity F , V_1 may or may not be a Liapunov function. If V_1 is not a Liapunov function, we inspect equation (27) and find a scalar function V_2 which is at least positive semidefinite and such that $(\dot{V}_1 + \dot{V}_2)$ is negative semidefinite. Thus, our Liapunov function is

$$V = V_1 + \int_0^t \dot{V}_2 dt. \quad (28)$$

The requirement which \dot{V}_2 fulfills is that it cancels that part of the right hand side of (27) which is not negative semidefinite.

In passing, we should note that for the nonlinear case Q need not always be nonsingular. This will occur if the system matrix \underline{A} has some pure imaginary characteristic values. The nonlinearities in the system then may produce a stable system, while the corresponding linear system is unstable.

WORK OF HALEY AND HARRIS

INTRODUCTION

The central theme of this work is the investigation of the time derivative of the V-function for the nonlinear system given in equation (2). By combining equations (8), (25) and (27), \dot{V} becomes

$$\dot{V} = \underline{x}_T \left[\underline{P} \underline{A} + \underline{A}_T \underline{P} \right] \underline{x} - 2 \underline{x}_T \underline{P} \underline{b} F, \quad (29)$$

where $\underline{P} = \underline{Q}_T \underline{Q}$. Harris discusses the various forms of the linear part of \dot{V} in reference [6], namely $\underline{x}_T \left[\underline{P} \underline{A} + \underline{A}_T \underline{P} \right] \underline{x}$. Haley's thesis, reference [7], deals with the constraints on \underline{Q} in the equation $\underline{Q} \underline{A} = \underline{R} \underline{Q}$ such that the nonlinear part of \dot{V} , $2 \underline{x}_T \underline{P} \underline{b} F$, is negative and semidefinite. Haley also considers complex transformations defined by the matrix \underline{Q} . Harris briefly discusses a Hurwitz Canonical Form applied to linear systems; but he finds that it contributes no appreciable additional stability information beyond that given by the Routh's Canonical Form.

HARRIS'S WORK

In reference [6], Harris considers the various \underline{R} and \underline{Q} matrices which can be developed for linear systems. For an n^{th} order system there exists 2^{n-1} independent \underline{Q} matrices and $n!$ different \underline{R} matrices. These various forms for \underline{R} and \underline{Q} do not all contribute new and significant information about the system being considered. For the third order case, Harris lists the \underline{R} and \underline{Q} matrices of Puri and Parks. In addition, Harris finds two other third order \underline{Q} matrices by integrating or differentiating certain elements of the corresponding differential equation and then multiplying the entire equation by this generated term. The resulting \underline{Q} 's produced by this method will be given in this section; but the application of the same technique to the direct generation of Liapunov functions will be discussed

in the Integration by Parts section of this report.

For a given Q matrix, Harris generated new R matrices by interchanging the rows in the Q matrix and then solving the equation $\underline{Q} \underline{A} = \underline{R} \underline{Q}$ for R. The reason for generating the various R matrices is to obtain different forms for the linear part of \dot{V} , namely:

$$y_T (\underline{R}_T + \underline{R}) y = 2 r_1 y_i^2, \quad (30)$$

where $i = 1, 2, \dots$, or n . As an example, listed below are the third order Q's and R's obtained by Puri and Harris:

PURI & WEYGANDT

$$\underline{Q} = \begin{bmatrix} \sqrt{r_2 r_3} & 0 & 0 \\ r_1 \sqrt{r_3} & \sqrt{r_3} & 0 \\ r_2 & r_1 & 1 \end{bmatrix}, \quad \underline{R} = \begin{bmatrix} -r_1 & \sqrt{r_2} & 0 \\ -\sqrt{r_2} & 0 & \sqrt{r_3} \\ 0 & -\sqrt{r_3} & 0 \end{bmatrix}, \quad (31)$$

$$(\dot{V})_{\text{LINEAR}} = -r_1 r_2 r_3 x_1^2.$$

HARRIS

$$\underline{Q} = \begin{bmatrix} 0 & \sqrt{r_2} & 0 \\ 0 & r_1 & 1 \\ r_1 \sqrt{r_3} & \sqrt{r_3} & 0 \end{bmatrix}, \quad \underline{R} = \begin{bmatrix} -r_1 & \sqrt{r_2} & 0 \\ -\sqrt{r_2} & 0 & -\sqrt{r_3} \\ 0 & \sqrt{r_3} & 0 \end{bmatrix}, \quad (32)$$

$$(\dot{V})_{\text{LINEAR}} = -r_1 r_2 x_2^2.$$

HARRIS

$$\underline{Q} = \begin{bmatrix} 0 & 0 & \sqrt{r_2} \\ a_3 & a_2 & 0 \\ 0 & r_1 \sqrt{r_3} & \sqrt{r_3} \end{bmatrix}, \quad \underline{R} = \begin{bmatrix} -r_1 & -\sqrt{r_2} & 0 \\ \sqrt{r_2} & 0 & \sqrt{r_3} \\ 0 & -\sqrt{r_3} & 0 \end{bmatrix}, \quad (33)$$

$$(\dot{V})_{\text{LINEAR}} = -r_1 r_2 x_3^2.$$

Harris noted that if one requires the linear part of \dot{V} to be negative definite and not just negative semidefinite, then the following procedure can be followed. Let r_1, r_2, \dots, r_n be the n elements in the n^{th} order \underline{R} matrix. Let $\underline{P}_i = \underline{Q}_i^T \underline{Q}_i$ be the matrix which satisfies

$$2 V_i = \underline{x}_T^T \underline{P}_i \underline{x},$$

and

$$\dot{V}_i = - r_1 r_2 \dots r_{n-1} x_i^2$$

where $i = 1, 2, \dots, n$, and where V_i is a Liapunov function. Now form a new Liapunov function by letting

$$2V = 2 (V_1 + V_2 + \dots + V_n),$$

where $\dot{V} = - (r_1 r_2 \dots r_{n-1}) \underline{x}_T^T \underline{x}$.

This \dot{V} is a negative definite function for a linear system if $[r_1 r_2 \dots r_n] > 0$.

The third order V whose derivative is negative definite can be formed from the \underline{Q} and \underline{R} matrices listed in the equations (31), (32), and (33).

HALEY'S WORK, 7

(a) Third Order - Real Transformation

We first consider the third order system where \underline{Q} is a real matrix. The \underline{A} and \underline{R} matrices are given as indicated below:

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \quad \text{and} \quad \underline{R} = \begin{bmatrix} -r & \sqrt{r_2} & 0 \\ -\sqrt{r_2} & 0 & \sqrt{r_3} \\ 0 & -\sqrt{r_3} & 0 \end{bmatrix}, \quad (34)$$

From the equation $\underline{Q}\underline{A} = \underline{R}\underline{Q}$ we get 9 equations involving the 9 unknown elements,

Q_{ij} , of \underline{Q} . Only 6 of these equations are independent and thus we have three degrees

of freedom. For example, if $Q_{13} = Q_{23} = 0$ and $Q_{31} = r_2$, we get the 3rd order Q -matrix of Puri and Weygandt. If $Q_{23} = 0$, $Q_{13} = 1$, and $Q_{31} = \sqrt{r_2 r_3}$, we get the Q -matrix of Parks.

Now consider the nonlinear part of \dot{V} , equation (29), $2\mathbf{x}_T \mathbf{P} \mathbf{b} F$. The only important terms in \mathbf{Pb} are the elements P_{13} , P_{23} and P_{33} . In other words, the nonlinear part of \dot{V} can be written as

$$2\mathbf{x}_T \mathbf{Pb} F = 2F \left[x_1 P_{13} + x_2 P_{23} + x_3 P_{33} \right]. \quad (35)$$

Haley considered the following six cases:

- Case 1: $P_{13} = 0$, F is a function of x_1 , x_2 and/or x_3 ,
- Case 2: $P_{13} = P_{23} = 0$, F is a function of x_2 and/or x_3 ,
- Case 3: $P_{23} = P_{33} = 0$, F is a function of x_1 and/or x_2 ,
- Case 4: $P_{13} = P_{33} = 0$, F is a function of x_1 , x_2 and/or x_3 ,
- Case 5: $P_{33} = 0$, F is a function of x_1 and/or x_2 ,
- Case 6: $P_{23} = 0$, F is a function of only x_2 .

The 6 independent equations from $\underline{QA} = \underline{RQ}$ plus the above restrictions allow us to find the Q matrices. Haley found for the above cases the following results: Cases 2,3,4 & 5 give only trivial solutions; case 6 is a special case of case 1. The results of case 1 are given in the following discussion.

The form of the nonlinear differential equation that can be analyzed is

$$\ddot{x} + a(x, \dot{x}, \ddot{x}) \ddot{x} + b(x, \dot{x}, \ddot{x}) \dot{x} + a_3 x = 0, \quad (36)$$

where the nonlinearities are greater than zero and

$$a(x, \dot{x}, \ddot{x}) = a_1 + a^1(x, \dot{x}, \ddot{x}),$$

$$b(x, \dot{x}, \ddot{x}) = a_2 + b^1(x, \dot{x}, \ddot{x}).$$

For asymptotic stability, the constants a_1 , a_2 , and a_3 must be positive, and $a_1 > 0$ and $b > 0$ for all x, \dot{x}, \ddot{x} , considered. The special cases where $a_1 = 0$, $a_2 = 0$, and/or $a_3 = 0$, will be considered in the examples at the end of this section.

The general form of F in equation (36) is

$$F = a^1 (x, \dot{x}, \ddot{x}) \ddot{x} + b (x, \dot{x}, \ddot{x}) \dot{x} \quad (37)$$

The transformation matrix Q can be written as:

$$Q = \begin{bmatrix} 0 & 0 & -\sqrt{r_2} \\ r_1 r_3 & r_2 + r_3 & 0 \\ 0 & r_1 \sqrt{r_3} & \sqrt{r_3} \end{bmatrix} \quad (38)$$

In state variable notation, the final Liapunov function and its time derivative are

$$\begin{aligned} V = & r_2 x_3^2 + [r_1 r_3 x_1 + (r_2 + r_3) x_2]^2 + \\ & + [r_1 \sqrt{r_3} x_2 + \sqrt{r_3} x_3]^2 + \\ & + 2 \int_{x_2(0)}^{x_2(t)} (r_2 + r_3) b^1(x_1, x_2, x_3) x_2 dx_2 + \\ & + 2 \int_{x_2(0)}^{x_2(t)} r_1 r_3 a^1(x_1, x_2, x_3) x_2 dx_2, \end{aligned} \quad (39)$$

and

$$\dot{V} = - [2r_1 r_2 - 2(r_2 + r_3) a^1(x_1, x_2, x_3)] x_3^2 - 2r_1 r_3 x_2^2 b^1(x_1, x_2, x_3). \quad (40)$$

The relationships between the r_i 's and the a_i 's are given by equation (14).

(B) Third Order Systems, Complex Q

The matrix Q in equation (8) is taken as complex, and thus the vector y is also complex. The candidate for a Liapunov function is given by

$$V = y^* T y = x_T^* Q_T^* Q x = x_T^* P^1 x, \quad (41)$$

where (*) indicates the complex conjugate. Haley studied the V - function in (41) and its corresponding time derivative, but the conclusion of his studies was that no significant new transformation, Q , can be found. This conclusion was found to be true for 2nd order and 4th order systems, as well as for 3rd order systems.

(c) Second Order System

For the second order system, the A and R matrices are given as

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \text{ and } \underline{R} = \begin{bmatrix} -r_1 & \sqrt{r_2} \\ \sqrt{r_2} & 0 \end{bmatrix}. \quad (42)$$

Following the same analysis as for the third order case, the corresponding Q matrix becomes

$$\underline{Q} = \begin{bmatrix} 0 & 1 \\ -\sqrt{r_2} & 0 \end{bmatrix}. \quad (43)$$

The most general nonlinear differential equation considered is

$$\ddot{x} + a_1 \dot{x} + a_2 x + F = 0, \quad (44)$$

where a_1 and a_2 are positive constants and $F = a^1(x, \dot{x}) \dot{x} + b^1(x, \dot{x}) x$. The

nonlinearities a^1 and b^1 are also positive. In state variable notation, the final form of the Liapunov function and its derivative is

$$V = r_2 x_1^2 + x_2^2 + 2 \int_{x_1(0)}^{x_1(t)} b^1(x_1, x_2) x_1 dx_1, \quad (45)$$

and

$$\dot{V} = -2r_1 x_2^2 - 2x_2 a^1(x_1, x_2). \quad (46)$$

(d) Fourth Order System

For the fourth order system the \underline{A} and \underline{R} matrices are

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix} \text{ and } \underline{R} = \begin{bmatrix} -r_1 & \sqrt{r_2} & 0 & 0 \\ -\sqrt{r_2} & 0 & \sqrt{r_3} & 0 \\ 0 & -\sqrt{r_3} & 0 & \sqrt{r_4} \\ 0 & 0 & -\sqrt{r_4} & 0 \end{bmatrix}. \quad (47)$$

As in the second and third order cases, the \underline{Q} is found to be

$$\underline{Q} = \begin{bmatrix} 0 & -\frac{r_1 + r_4}{r_2} & 0 & -1/\sqrt{r_2} \\ r_4 & 0 & 1 & 0 \\ 0 & -\sqrt{r_3} & 0 & 0 \\ \sqrt{r_3 r_4} & 0 & 0 & 0 \end{bmatrix} \quad (48)$$

The form of the differential equation considered here is

$$\ddot{\ddot{x}} + a_1 \ddot{\ddot{x}} + a_2 \ddot{\ddot{x}} + a_3 \dot{\ddot{x}} + a_4 \dot{x} + F = 0, \quad (49)$$

where a_1, a_2, a_3 and a_4 are positive constants and $F = b^1(x, \dot{x}, \ddot{x}, \ddot{\ddot{x}}) \ddot{x}$. The nonlinearity b^1 is also positive. In state variable notation, the final form of the Liapunov function and its derivative is

$$\begin{aligned}
 V = & \left[\frac{a_3}{r_1 \sqrt{r_2}} x_2 + \frac{1}{\sqrt{r_2}} x_4 \right]^2 + (r_4 x_1 + x_3)^2 + \\
 & + r_3 x_2^2 + r_3 r_4 x_1^2 + 2 \int_{x_2(0)}^{x_2(t)} \left[\frac{a b^1}{r_1 r_2} \right] x_2 dx_2 + \\
 & + 2 \int_{x_3(0)}^{x_3(t)} \left[\frac{1}{r_2} \right] x_3 dx_3,
 \end{aligned} \tag{50}$$

and

$$\dot{V} = -2r_1 \left[\frac{a_3}{r_1 \sqrt{r_2}} x_2 + \frac{1}{\sqrt{r_2}} x_4 \right]^2. \tag{51}$$

HARRIS'S "HURWITZ CANONICAL FORM"

In reference [6], Harris derived a Hurwitz Canonical Form to be used in stability analysis. For second order cases the Routh and Hurwitz Canonical Forms give equivalent Liapunov functions. This is not true for third and higher order systems. According to Harris, the Liapunov functions given by this canonical form are not as useful as those given by the Routh Canonical Form. Thus, we will only briefly outline Harris's derivation of the Hurwitz Canonical Form.

Rewrite equation (1) in the following way

$$\ddot{x} + \frac{a_{n-1}}{a_n} \dot{x}^{(1)} + \frac{a_{n-2}}{a_n} \dot{x}^{(2)} + \dots + \frac{x^{(n)}}{a_n} + \frac{F(\underline{x}, t)}{a_n} = 0, \quad (52)$$

or

$$\ddot{x} + \alpha_1 \dot{x}^{(1)} + \alpha_2 \dot{x}^{(2)} + \dots + \alpha_n \dot{x}^{(n)} + F^1(\underline{x}, t) = 0,$$

where $\alpha_1, \dots, \alpha_n$ are constants and $F^1(\underline{x}, t)$ is the nonlinear part such that $F^1(0, t) = 0$.

Define the state variable \underline{x} by

$$x_1 = x^{(n)}, \quad (53)$$

$$x_2 = x^{(n-1)},$$

.....

$$x_n = x^{(1)},$$

$$\int \ddot{x}_n dt = \dot{x} = -\alpha_n x_1 - \alpha_{n-1} x_2 - \dots - \alpha_1 x_n - F^1(\underline{x}, t).$$

In matrix form (52) becomes

$$\int \underline{\ddot{x}} dt = \underline{A} \underline{x} - \underline{b} F^1 \quad (54)$$

where $\underline{A} =$

0	1	0	...	0	0
0	0	1	...	0	0
.
.
0	0	0	...	0	1
$-\alpha_n$	$-\alpha_{n-1}$	$-\alpha_{n-2}$...	$-\alpha_2$	$-\alpha_1$

and $\underline{b} =$

0
0
.
.
0
1

(55)

Following the same procedure as in the derivation of the Routh Canonical Form, a change of variable \underline{y} is defined by

$$\underline{y} = \underline{Q} \underline{x}, \quad (56)$$

where \underline{Q} is a constant matrix. Substituting (56) into (54) gives

$$\int \underline{y} \, dt = \underline{R} \underline{y} - \underline{Q} \underline{b} \underline{F}^1(\underline{x}, t) \quad (57)$$

where $\underline{R} = \underline{Q} \underline{A} \underline{Q}^{-1}$. Now let the candidate for a Liapunov function be

$$2V = \left(\int \underline{y}_T dt \right) \left(\int \underline{y} \, dt \right), \quad (58)$$

and the corresponding time derivation is

$$\begin{aligned} 2\dot{V} &= \left(\int \underline{y}_T \, dt \right) \underline{y} + \underline{y}_T \left(\int \underline{y} \, dt \right) \\ &= \underline{y}_T (\underline{R}_T + \underline{R}) \underline{y} - 2\underline{y}_T \underline{Q} \underline{b} \underline{F}^1(\underline{x}, t). \end{aligned} \quad (59)$$

Equation (59) indicates that we may use all the previous formulas of the Routh Canonical Form technique to generate these new Liapunov functions. All that is necessary is to replace the \underline{a} 's in the \underline{Q} matrices by α 's and replace the \underline{r} 's in the \underline{R} matrices by β 's. The form of the conditions for stability remain the same and the conditions for stability in linear systems are the same. The β 's in the new \underline{R} matrices are related to the Hurwitz determinants. The major difficulty of this method is that the nonlinear term in \dot{V} , $\underline{y} \underline{Q} \underline{b} \underline{F}'$, is not as simple as that given by the Routh Canonical Form.

Conclusions

In this section we have discussed the Routh Canonical Form and its relationship to the stability of a nonlinear system. First, the work of Puri and Weygandt was discussed and their method for generating Liapunov functions was described. The work of Haley and Harris resulted in the analysis of rather general second, third, and fourth order nonlinear, autonomous differential equations. The method can be extended to any order equation; but, as the order increases, the labor becomes prohibitive due to the various \underline{R} and \underline{Q} matrices which are possible in this analysis.

COMPENDIUM OF EXAMPLES

Second Order Examples

Example 1, [5] & [6]

The system is described by

$$\ddot{x} + \dot{x} + kx^3 = 0, \quad k = \text{constant}.$$

Writing this equation in state variable form gives:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} k x_1^3,$$

where $x_1 = x$, $x_2 = \dot{x}$.

Since $r_1 = a_1 = -1$ and $r_2 = a_2 = 0$, then

$$Q = \begin{bmatrix} \sqrt{r_2} & 0 \\ r_1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \det. Q = 0.$$

(We note that Q is singular in this example.)

The transformation $y = Q x$ becomes

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 + x_2 \end{bmatrix}.$$

The V_1 -function defined by $y_T y$ is

$$V_1 = y_T y = (x_1 + x_2)^2,$$

where

$$\dot{V}_1 = -2K (x_1 + x_2) x_1^3 = -2K x_1^4 - 2K x_2 x_1^3$$

Add $\dot{V}_2 = 2Kx_2x_1^3$ to \dot{V}_1 in order that the sum is a semi-definite form. Therefore,

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -2Kx_1^4,$$

and

$$V = (x_1 + x_2)^2 + 2K \int_0^t x_1^3 x_2 dt = (x_1 + x_2)^2 + \frac{Kx_1^4}{2}.$$

Thus, V is positive definite and \dot{V} is negative semi-definite if $K > 0$. Since $V \rightarrow \infty$ as $\|\underline{x}\| \rightarrow \infty$ and since no trajectory of the system makes \dot{V} identically zero except the trivial solution, the system is globally asymptotically stable.

Example 2, [7]

From the field of electronencephalography, we have

$$\ddot{x} + (a + bx^2 - cx^6 + dx^{10}) \dot{x} + x = 0$$

where a, b, c and d are positive constants. Using state variables, we have:

$$\dot{x}_1 = x_2 = \dot{x}$$

$$\dot{x}_2 = ax_2 - x_1 - f$$

$$f = (-2a + bx_1^2 - cx_1^6 + dx_1^{10}) x_2$$

Haley's Q matrix takes the form

$$Q = \begin{bmatrix} 0 & 1 \\ -\sqrt{r_2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $r_1 = a$ and $r_2 = 1$. Thus, V and \dot{V} become

$$V = \underline{y}_T \underline{y} = x_1^2 + x_2^2$$

$$\dot{V} = \underline{x}_T \left[\underline{A}_T Q_T Q + Q_T Q \underline{A} \right] \underline{x} - 2\underline{x}_T Q_T Q \underline{b} \cdot f$$

$$= 2a x_2^2 - 2x_2^2 (-2a + bx_1^2 - cx_1^6 + dx_1^{10}).$$

V is positive definite and \dot{V} is negative semi-definite when

$$b x_1^2 + dx_1^{10} > -2a + cx_1^6.$$

When this inequality is satisfied the nonlinear differential equation is asymptotically stable.

Example 3, [7]

Consider the equation:

$$\ddot{x} + a\dot{x} + b^2 f(x) = 0,$$

where $f(x)$ is defined by

$$f(x) = x - f_1(x),$$

$$f_1(x) = 0, \quad -K \leq x \leq K,$$

$$f_1(x) = x - K, \quad x > K,$$

$$f_1(x) = x + K, \quad x < -K.$$

Rewriting the differential equation gives

$$\ddot{x} + a\dot{x} + b^2 x = b^2 f_1(x),$$

and in state variables

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -ax_2 - b^2 x_1 + b^2 f_1(x_1).$$

The Q matrix becomes

$$Q = \begin{bmatrix} 0 & 1 \\ -b & 0 \end{bmatrix},$$

where $r_1 = a$, $r_2 = b^2$, and $b > 0$. Thus,

$$V_1 = \underline{y}^T \underline{y} = x_2^2 + b^2 x_1^2$$

$$\dot{V}_1 = -2a x_2^2 + 2b^2 f_1(x_1) x_2.$$

Adding $\dot{V}_2 = -2b^2 f_1(x_1) x_2$ to \dot{V}_1 gives

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -2ax_2^2$$

$$V = b^2 x_1^2 + x_2^2 - 2b^2 \int_{x_1(0)}^{x_1(t)} f_1(x_1) dx_1.$$

It can be shown that if $a > 0$ and $-K < x_1 < K$, then the system is asymptotically stable.

The system will also be asymptotically stable whenever $x_1(t)$ and $x_1(0)$ satisfy

$$x_1^2 > 2 \int_{x_1(0)}^{x_1(t)} f_1(x_1) dx_1.$$

Example 4, [5]

This example was analyzed by Schultz and Gibson using the variable gradient method.

The equation is

$$\ddot{x} + \dot{x} + f(x) \dot{x} + \frac{df(x)}{dx} x \dot{x} + \beta x f(x) = 0.$$

In state variable notation we have:

$$x_1 = x, \quad x_2 = \dot{x},$$

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -x_2 - F(x_1, x_2),$$

$$F(x_1, x_2) = f(x_1) x_2 + \frac{df(x_1)}{dx_1} x_1 x_2 + \beta x_1 f(x_1),$$

$$\underline{A} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} F(x_1, x_2).$$

The Q matrix is

$$Q = \begin{bmatrix} \sqrt{r_2} & 0 \\ r_1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

where $r_1 = a_1 = 1$ and $r_2 = a_2 = 0$. (Note, this is another example where Q is singular.) The V_1 function is

$$V_1 = \underline{y}_T \underline{y} = (x_1 + x_2)^2,$$

where

$$\begin{aligned} \dot{V}_1 &= -2r_1 y_1^2 - 2y_2 F \\ &= -2(x_1 + x_2) \left[f(x_1) x_2 + \frac{df(x_1)}{dx_1} x_1 x_2 + \theta x_1 f(x_1) \right]. \end{aligned}$$

Now let \dot{V}_2 be

$$\dot{V}_2 = 2x_1 x_2 \left[\frac{dx_1 f(x_1)}{dx_1} + \theta f(x_1) \right].$$

Thus, $\dot{V}_1 + \dot{V}_2$ becomes

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -2\theta x_1^2 f(x_1) - 2x_2^2 \left[\frac{dx_1 f(x_1)}{dx_1} \right]$$

$$\text{and } V = (x_1 + x_2)^2 + 2 \int_0^{x_1} \left[\frac{dx_1 f(x_1)}{dx_1} + \theta f(x_1) \right] x_1 dx_1.$$

Therefore, the system is globally asymptotically stable if:

- (1) $\frac{dx_1 f(x_1)}{dx_1} > 0,$
- (2) $\theta f(x_1) > 0,$
- (3) $\dot{V} \neq 0$ on any nontrivial trajectory,
- (4) $V \rightarrow \infty$ if $\|\underline{x}\| \rightarrow \infty.$

Third Order Examples

Example 1, [5]

Consider the following system

$$\ddot{x} + a\dot{x} + bx + abx + F = 0$$

where a and b are positive constants, and F is a nonlinearity which makes the unstable linear system stable by its presence. The Q matrix is

$$Q = \begin{bmatrix} \sqrt{r_1 r_3} & 0 & 0 \\ r_1 \sqrt{r_3} & \sqrt{r_3} & 0 \\ r_2 & r_1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a\sqrt{b} & \sqrt{b} & 0 \\ 0 & a & 1 \end{bmatrix},$$

where $r_1 = a_1 = a$, $r_2 = a_2 - \frac{a_3}{a_1} = b - \frac{ab}{a} = 0$, and $r_3 = a_3/a_1 = ab/a = b$. Thus

the V_1 -function becomes

$$V_1 = \underline{y}_T \underline{y} = b (ax_1 + x_2)^2 + (ax_2 + x_3)^2,$$

where $x_1 = x$, $x_2 = \dot{x}$, and $x_3 = \ddot{x}$. The time derivative of V_1 is

$$\dot{V}_1 = -2ay_1^2 - 2Fy_3 = -2(ax_2 + x_3)^2 F.$$

Choose $F = K x_2^{2m+1}$, where m is a nonnegative integer. Thus, \dot{V}_1 becomes

$$\dot{V}_1 = -2aKx_2^{2(m+1)} - 2Kx_3 x_2^{2(m+1)}$$

This dictates the choice of \dot{V}_2 , namely

$$\dot{V}_2 = 2Kx_3 x_2^{2m+1}$$

Thus,

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -2aK x_2^{2(m+1)},$$

and

$$V = b(ax_1 + x_2)^2 + (ax_2 + x_3)^2 + \frac{K}{m+1} x_2^{2m+2}$$

The system is globally asymptotically stable if m is nonnegative integer, $K > 0$, $a > 0$, $b > 0$, and $F = K x_2^{2m+1}$.

Example 2, 5

The following example comes from the Russian author, E. P. Popov:

$$\ddot{x} + \frac{1}{T} \dot{x} + \frac{K}{T} F(\alpha) = 0,$$

where

$$\alpha = K_2 x + K_3 \dot{x} - K_4 \int_0^t F(\alpha) dt,$$

and T, K_1, K_2, K_3, K_4 are constants. In state variable notation, we have

$$\dot{x}_1 = x_2 = \dot{x}$$

$$\dot{x}_2 = x_3 = \dot{\dot{x}}$$

$$\dot{x}_3 = -\frac{1}{T} x_3 - F_1, \quad F_1 = \frac{K}{T} F(\alpha).$$

Since

$$a_1 = 1/T = r_1, \quad a_2 = a_3 = 0 \text{ and } r_2 = r_3 = 0,$$

The Q matrix becomes

$$Q = \begin{bmatrix} r_2 \sqrt{r_3} & 0 & 0 \\ r_1 \sqrt{r_1} & \sqrt{r_3} & 0 \\ r_2 & r_1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/T & 1 \end{bmatrix}.$$

The V_1 - function is given as

$$V_1 = \underline{y}_T \underline{y} = \left(\frac{x_2}{T} + x_3 \right)^2,$$

where

$$\dot{V}_1 = -\frac{2k}{T} \left(\frac{x_2}{T} + x_3 \right) F(\alpha).$$

The time derivative of α is given by

$$\dot{\alpha} = K_2 X_2 + K_3 X_3 - K_4 F(\alpha).$$

From the original differential equation and the above expression for $\dot{\alpha}$ we can finally obtain

$$\begin{aligned} \frac{2K_1}{T^2} X_2 F(\alpha) &= \frac{2K_1}{K_2 T^2} F(\alpha) \dot{\alpha} + \frac{2K_3}{K_2 T} X_3^2 + \\ &+ \frac{K_3}{K_2 T} X_3 \dot{X}_3 + \frac{2 K_1 K_4}{T^2} F^2(\alpha). \end{aligned}$$

Substitute this expression into the equation for \dot{V}_1 and then form \dot{V}_2 ; that is

$$\dot{V}_2 = \frac{2K_1}{K_2 T^2} F(\alpha) \dot{\alpha} + 2 \left(\frac{K_3}{K_2 T} - 1 \right) X_3 \dot{X}_3.$$

Hence, we have

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -\frac{2K_1}{T} \left[\left(\frac{K_3}{K_2 T} - 1 \right) \frac{X_3^2 T}{K_1} + \frac{K_4}{K_2 T} F^2(\alpha) \right],$$

and

$$V = \left(\frac{1}{T} X_2 + X_3 \right)^2 + \frac{2 K_1}{K_2 T^2} \int_0^d F(\alpha) d\alpha + \left(\frac{K_3}{K_2 T} - 1 \right) X_3^2.$$

The conditions for asymptotic stability are:

$$(1) \quad K_1 > 0, K_2 > 0, K_4 > 0,$$

$$(2) \quad \left(\frac{K_3}{K_2 T} - 1 \right) > 0,$$

$$(3) \quad T > 0,$$

$$(4) \quad \int_0^d F(\alpha) d\alpha > 0.$$

Example 3, [6]

Harris's example is

$$\ddot{x} + a_1 \ddot{x} + a_2 \dot{x} + a_3 x^3 = 0,$$

where $r_1 = a_1$, $r_2 = a_2$, $r_3 = 0$ and

$F(x, t) = a_3 x^3$. The V_1 - function becomes

$$2V_1 = a_2 \dot{x}^2 + (a_1 \dot{x} + \ddot{x})^2,$$

where

$$\dot{V}_1 = -a_1 a_2 \dot{x}^2 - (a_1 \dot{x} + \ddot{x})(a_3 x^3).$$

Define \dot{V}_2 as

$$\dot{V}_2 = a_1 a_3 x^3 \dot{x} + a_3 x^3 \ddot{x} + 3a_3 x^2 \dot{x}^2.$$

Thus, we have

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -a_1 (a_2 - 3a_3/a_1 x^2) \dot{x}^2,$$

$$2V = a_2 \dot{x}^2 + (a_1 \dot{x} + \ddot{x})^2 + a_1 a_3 / 2 x^4 + a_3 x^3 \dot{x},$$

$$= \left[a_2 - \frac{a_3}{2a_1} x^2 \right] \dot{x}^2 + \left[a_1 \dot{x} + \ddot{x} \right]^2 + \frac{a_3}{2a_1} x^2 (a_1 \dot{x} + \ddot{x})^2.$$

The conditions for asymptotic stability are

$$(1) \quad a_1 > 0, a_2 > 0, a_3 > 0,$$

$$(2) \quad \frac{a_1 a_2}{3 a_3} > x^2.$$

Example 4, [7]

The differential equation is

$$\ddot{x} + a \ddot{x} + b(\dot{x}) \dot{x} + c x = 0.$$

Using state variables and writing $b(\dot{x}) \dot{x}$ as $b(\dot{x}) \dot{x} = b \dot{x} + b'(\dot{x}) \dot{x}$,

b being a constant, we get:

$$\dot{x}_1 = x_2 = \dot{x},$$

$$\dot{x}_2 = x_3,$$

$$\dot{x}_3 = -a x_3 - b x_2 - c x_1 - b'(x_2) x_2.$$

Haley's Q matrix is

$$Q = \begin{array}{|c|c|c|} \hline 0 & 0 & -\sqrt{r_2} \\ \hline r_1 r_3 & r_2 + r_3 & 0 \\ \hline 0 & r_1 \sqrt{r_3} & \sqrt{r_3} \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 0 & 0 & -\sqrt{b-c/a} \\ \hline c & b & 0 \\ \hline 0 & a\sqrt{c/a} & \sqrt{c/a} \\ \hline \end{array}$$

where $r_1 = a$, $r_2 = b - c/a$ and $r_3 = c/a$.

Thus, the V_1 - function becomes

$$V_1 = \underline{Y}^T \underline{Y} = (b - c/a) x_3^2 + (c x_1 + b x_2)^2 + \left[\sqrt{ac} x_2 + \sqrt{c/a} x_3 \right]^2,$$

where

$$\dot{V}_1 = -2a(b - c/a) x_3^2 - 2b'(x_2) x_2 [c x_2 + b x_3].$$

Let \dot{V}_2 be defined by

$$\dot{V}_2 = 2bb'(x_2) x_2 x_3;$$

then

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -2(ab-c) x_3^2 - 2cb'(x_2) x_2^2,$$

and

$$V = (b - c/a) x_3^2 + (c x_1 + b x_2)^2 + \left[\sqrt{ac} x_2 + \sqrt{c/a} x_3 \right]^2 + 2 \int_{x_2(0)}^{x_2(t)} b b'(x_2) x_2 dx_2.$$

This is our Liapunov function. Thus, for asymptotic stability we require

- (1) $b'(x_2) > 0$, $b > 0$,
- (2) $a > 0$, $c > 0$,
- (3) $ab - c > 0$.

Example 5, [7]

The stability of the null solution of a system, characterized by the following differential equation, is investigated:

$$\ddot{X} + f(x, \dot{x}) \ddot{x} + b \dot{x} + cx = 0$$

where f is written as

$$f(x, \dot{x}) = a + a' (x, \dot{x}).$$

The differential equation in state variable form is

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -ax_3 - bx_2 - cx_1 - a'(x_1, x_2)x_3. \end{aligned}$$

The transformation Q , Haley's form, is

$$Q = \begin{array}{|c|c|c|} \hline 0 & 0 & -\sqrt{b - c/a} \\ \hline c & b & 0 \\ \hline 0 & a\sqrt{c/a} & \sqrt{c/a} \\ \hline \end{array}$$

where $r_1 = a$, $r_2 = b - c/a$ and $r_3 = c/a$.

The V_1 - function becomes

$$V_1 = \underline{Y}_t \underline{Y} = (b - c/a) X_3^2 + (cx_1 + bx_2)^2 + c/a (ax_2 + x_3)^2,$$

where $\dot{V}_1 = -2(ab-c) X_3^2 - 2a'(X_1, X_2) X_3 (cx_2 + bx_3)$.

Thus, let \dot{V}_2 be

$$\dot{V}_2 = 2 ca' (x_1, x_2) x_2 x_3.$$

Therefore, the resulting \dot{V} and V are

$$\begin{aligned}\dot{V} &= \dot{V}_1 + \dot{V}_2 = -2 x_3^2 \left[ab + ba' (x_1, x_2) - c \right] \\ &= -2 \left[bf (x_1, x_2) - c \right] x_3^2,\end{aligned}$$

and

$$\begin{aligned}V &= (b-c/a) x_3^2 + (cx_1 + bx_2)^2 + c/a (ax_2 + x_3)^2 + \\ &+ 2c \int_0^{x_2} a'(x_1, x_2) x_2 dx_2.\end{aligned}$$

The conditions which must be satisfied for asymptotic stability are:

- (1) $bf (x_1, x_2) - c > 0$,
- (2) $a > 0, c > 0, ab-c > 0$,
- (3) $\int_0^{x_2} a' (x_1, x_2) x_2 dx_2 > 0$.

Example 6, [7]

The differential equation is

$$\ddot{X} + A(x, \dot{x}) \ddot{x} + B(x, \dot{x}) \dot{x} + cx = 0.$$

In state variable notation,

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3,$$

$$\dot{x}_3 = -ax_3 - bx_2 - cx_1 - f,$$

where

$$A(x_1, x_2) = a + a'(x_1, x_2),$$

$$B(x_1, x_2) = b + b'(x_1, x_2),$$

$$f = a'(x_1, x_2) x_3 + b'(x_1, x_2) x_2.$$

Fourth Order Example, 7

This example has been discussed by Cartwright but will be considered here in the "light" of the Routh Canonical Form:

$$\ddot{\ddot{x}} + a_1 \ddot{x} + f(\dot{x}) \ddot{x} + a_3 \dot{x} + a_4 x = 0,$$

where

$$f(\dot{x}) = a_2 + b'(\dot{x}).$$

In state variable notation we have

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3,$$

$$\dot{x}_3 = x_4,$$

$$\dot{x}_4 = -a_1 x_4 - a_2 x_3 - a_3 x_2 - a_4 x_1 - f,$$

$$f = b'(x_2) x_3.$$

Since we have

$$r_1 = a_1,$$

$$r_2 = [a_2 - a_3/a_1],$$

$$r_3 = [a_3/a_1 - a_1 a_4/(a_1 a_2 - a_3)],$$

$$r = [a_1 a_4/(a_1 a_2 - a_3)],$$

then the Q matrix becomes

$$Q = \begin{array}{c} \begin{array}{|c|c|c|c|} \hline 0 & -\frac{a_3}{a_1} \sqrt{\frac{a_1}{a_1 a_2 - a_3}} & 0 & -\sqrt{\frac{a_1}{a_1 a_2 - a_3}} \\ \hline \frac{a_1 a_4}{a_1 a_2 - a_3} & 0 & 1 & 0 \\ \hline 0 & -\sqrt{\frac{a_3}{a_1} - \frac{a_1 a_4}{a_1 a_2 - a_3}} & 0 & 0 \\ \hline \sqrt{r_3 r_4} & 0 & 0 & 0 \\ \hline \end{array} \end{array}$$

Thus, the V_1 - function is

$$\begin{aligned} V_1 &= \underline{Y}_t \underline{Y} = Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 \\ &= \left[\frac{a_3}{a_1} \sqrt{\frac{a_1}{a_1 a_2 - a_3}} x_2 + \sqrt{\frac{a_1}{a_1 a_2 - a_3}} x_4 + \frac{a_1 a_4}{a_1 a_2 - a_3} x_1 + x_3 \right]^2 \\ &\quad + \left[\frac{a_3}{a_1} - \frac{a_1 a_4}{a_1 a_2 - a_3} \right] x_2^2 + \left[\frac{a_1 a_4}{a_1 a_2 - a_3} \right] \left[\frac{a_3}{a_1} - \frac{a_1 a_4}{a_1 a_2 - a_3} \right] x_1^2, \end{aligned}$$

where

$$\dot{V}_1 = -2a_1 \left[\frac{a_3 x_2}{a_1 a_2 - a_3} + \frac{a_1 x_4}{a_1 a_2 - a_3} \right]^2 - 2b'(x_2) x_3 \left(\text{times} \right) \left[\frac{a_3 x_2 + a_1 x_4}{a_1 a_2 - a_3} \right].$$

Therefore, we let \dot{V}_2 be given by

$$\dot{V}_2 = 2b'(x_2) x_3 \left[\frac{a_3 x_2 + a_1 x_4}{a_1 a_2 - a_3} \right].$$

The final Liapunov function is

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -2a_1 \left[\frac{a_3 x_2 + a_1 x_4}{a_1 a_2 - a_3} \right]^2,$$

and

$$V = V_1 + \frac{2a_3}{a_1 a_2 - a_3} \int_0^{x_2} b'(x_2) x_2 dx_2 + \frac{2a_1}{a_1 a_2 - a_3} \int_0^{x_3} b'(x_2) x_3 dx_3.$$

The condition for asymptotic stability are:

The conditions for asymptotic stability are:

$$(1) \quad a_1 > 0, \quad a_1 a_4 > 0, \quad (a_1 a_2 - a_3) > 0, \quad a_3 > 0,$$

$$(2) \quad a_3/a_1 - \frac{a_1 a_4}{a_1 a_2 - a_3} > 0,$$

$$(3) \quad \int_0^{x_2} b'(x_2) x_2 dx_2 > 0,$$

$$(4) \quad \int_0^{x_3} b'(x_2) x_3 dx_3 > 0.$$

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SECTION FOUR

INTEGRATION BY PARTS

Prepared by: R. L. Drake
T. Gifford

INTEGRATION BY PARTS

SUMMARY

In this section we discuss the generation of Liapunov functions by a repeated application of integration by parts. We discuss three variations of this procedure, as given by Geiss and Reiss, Harris and Puri. The method of Geiss and Reiss begins with a first approximation for \dot{V} ; for n^{th} order systems, $\dot{V}_1 = -x_n^2$. By integration by parts, V_1 is evaluated. During the application of the integration by parts, the system equations play the role of constraints along the paths of integration. If V_1 is not a Liapunov function, a second approximation for \dot{V} is considered, namely, $\dot{V}_2 = -(x_n^2 + \alpha x_{n-1}^2)$. The constant α is chosen in such a way that V_2 is definite, if this is possible. By integration by parts, V_2 is obtained. If V_2 is not a Liapunov function, then choose

$$\dot{V}_3 = -(x_n^2 + \alpha x_{n-1}^2 + \beta x_{n-2}^2).$$

V_3 is obtained by integrating \dot{V}_3 . The procedure continues until a Liapunov function is given or the method fails. This method works best for low order systems.

In Harris' work, multipliers are formed by differentiating or integrating certain terms in the given differential equation. These multipliers are then applied to the original differential equation. The result from this operation is that certain terms in the resulting equation are perfect squares in the state variables, and thus form \dot{V} . Other terms are time derivatives of perfect squares, and still other terms are reformulated by the integration by parts technique such that \dot{V} can be integrated to give a positive definite V - function. This method is applied to second and third order systems.

Puri's method is basically a combination of Harris' method and the Geiss and Reiss method. Puri's method is more systemetized and can probably be applied to higher order systems with more ease than the other two methods. Puri's technique involves both a multiplier and repeated integration by parts.

INTRODUCTION

We are concerned with the study of stability of the equilibrium solution (the origin in state space) of a dynamic system defined by

$$\dot{\underline{x}} = \underline{F}(\underline{x}, t) . \quad (1)$$

The technique considered here for generating Liapunov functions can be applied to certain nonautonomous systems as well as to autonomous systems, look at examples 5 and 17 in this section.

First, we will discuss the repeated integration by parts technique of Geiss and Reiss [1,2,3]*. And then the method of Harris [4], will be considered. This discussion deals with the application of Harris' technique applied to a third order linear system, with constant coefficients. The extension of Harris' method to equations with one nonlinearity is given in examples 6 through 12.

Puri's method, [5], will be discussed in more detail than the other two techniques because it is more systematic. In the compendium of examples, the applicability of this method is exhibited. Puri's method was used in reference [6] to calculate quadratic moments of high order linear systems.

WORK OF GEISS & REISS

This procedure is a simple application of integration by parts to the problem of obtaining Liapunov functions for ordinary differential equations. The autonomous nonlinear system which is considered is given by

$$\dot{\underline{x}} = \underline{f}(\underline{x}) , \quad \underline{f}(\underline{0}) = \underline{0} . \quad (2)$$

We want to find a positive semidefinite form $\mathcal{V}(\underline{x})$, such that

$$\dot{\mathcal{V}} = (\nabla \mathcal{V})_t \underline{f}(\underline{x}) = - \mathcal{V}(\underline{x}) , \quad (3)$$

* The numbers in the brackets [] refer to the references at the end of the section.

and

$$V = \int \dot{V} dt = - \int \mathcal{V}(\underline{x}) dt, \quad (4)$$

where V is positive definite. The integral in (4) is evaluated by integration by parts. As we know, the integration by parts formula involving two functions of a single variable is given by

$$\int u(t) dv(t) = u(t)v(t) - \int v(t) du(t). \quad (5)$$

The technique of Geiss and Reiss is concerned with the choice of $\mathcal{V}(\underline{x})$ in equation (3). For an n^{th} order system, their first approximation for $\mathcal{V}(\underline{x})$ is

$$\mathcal{V}_1(\underline{x}) = x_n^2, \quad (6)$$

where the usual state variable notation is assumed to be used. The corresponding V - function is

$$V_1 = \int (-x_n^2) dt \quad (7)$$

where (7) is evaluated by using integration by parts and considering the system equation (2) as a constraint along the path of integration. If V_1 is a Liapunov function, we stop the process. If V_1 is not a Liapunov function, we continue with the following second approximation:

$$\dot{V}_2 = - (x_n^2 + \alpha x_{n-1}^2), \quad (8)$$

thus

$$V_2 = - \int x_n^2 dt - \alpha \int x_{n-1}^2 dt, \quad (9)$$

where the integrals in (9) are evaluated by integration by parts and α is an arbitrary constant used to make V_2 positive definite, if possible. If V_2 fails to be a Liapunov function, the third approximation is

$$\dot{V}_3 = - (x_n^2 + \alpha x_{n-1}^2 + \beta x_{n-2}^2). \quad (10)$$

The procedure follows this pattern until a Liapunov function is found, or the method fails.

The disadvantages of this method are:

- (1) it is limited to quasilinear systems of low order,
- (2) the vector function $\underline{f}(\underline{x})$ in (2) must be such that it can be solved for one of the state variables.

The advantages of the method are:

- (1) The simplicity of the method,
- (2) it gives insight into the construction of Liapunov functions,
- (3) it can be adopted to handle equations containing arbitrary functions of one or more state variables,
- (4) it can be used to modify an existing Liapunov function, example 1,
- (5) it is useful in the construction of instability proofs and in considering the concept of complete stability, and
- (6) for linear systems, the method gives the Routh-Hurwitz conditions of stability.

Let us consider a problem of instability. For example, consider a third order system. We select a \dot{V} of the form

$$\dot{V} = c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2, \quad (11)$$

where c_1, c_2, c_3 are positive constants. Thus \dot{V} is positive definite. By integration by parts, we obtain

$$V = f_1(x_1, x_2, x_3; c_1, c_2, c_3). \quad (12)$$

If it is possible to choose the positive constants, C_i , such that there are points arbitrarily close to $\underline{x} = \underline{0}$ (equilibrium solution) where V takes on positive values, then the system is unstable at $\underline{x} = \underline{0}$.

Examples of the work of Geiss and Reiss are given at the end of this section.

WORK OF HARRIS

The method of Harris uses integration by parts after applying certain multipliers to the original differential equation. This procedure is applicable for low order systems. In this discussion, we will apply the method to a third order linear system. In this way we can demonstrate the mechanics of the technique, the reasoning behind the technique, and how the technique can be applied to nonlinear systems. The applicability of the method to nonlinear equations is considered in examples 6 through 12 at the end of this section.

Consider the third order linear, time - invariant system defined by

$$\ddot{x} + a_1 \ddot{x} + a_2 \dot{x} + a_3 x = 0. \quad (13)$$

Multiply (13) by the integral of the first two terms in (13), namely:

$$\int (\ddot{x} + a_1 \ddot{x}) dt = \dot{x} + a_1 x. \quad (14)$$

Thus, we have

$$(\dot{x} + a_1 x) (\ddot{x} + a_1 \ddot{x}) + a_2 \dot{x} \ddot{x} + a_1 a_3 x \dot{x} + a_1 a_2 (\dot{x})^2 + a_3 x \ddot{x} = 0. \quad (15)$$

Rewriting (15) gives

$$\frac{1}{2} \frac{d}{dt} \left[(\ddot{x} + a_1 \dot{x})^2 + a_2 (\dot{x})^2 + a_1 a_3 x^2 \right] + a_3 x \ddot{x} = - a_1 a_2 (\dot{x})^2. \quad (16)$$

Applying integration by parts to $a_3 x \ddot{x}$, gives

$$a_3 x \ddot{x} = a_3 \frac{d(x \dot{x})}{dt} - a_3 (\dot{x})^2. \quad (17)$$

Combining (16) and (17), we get:

$$\frac{1}{2} \frac{d}{dt} \left[(\ddot{x} + a_1 \dot{x})^2 + a_2 (\dot{x})^2 + a_1 a_3 x^2 + 2 a_3 x \dot{x} \right] = - (a_1 a_2 - a_3) (\dot{x})^2. \quad (18)$$

Now, define \dot{V} to be equal to the right-hand side of (18), namely

$$\dot{V} = - (a_1 a_2 - a_3) (\dot{x})^2. \quad (19)$$

Thus, from (18) and (19), we can derive the expression for V,

$$2V = (\ddot{x} + a_1\dot{x})^2 + (a_2 - a_3/a_1)(\dot{x})^2 + a_1/a_3 (a_1x + \dot{x})^2. \quad (20)$$

For asymptotic stability, a Liapunov function defined by (19) and (20) implies the following conditions:

$$(1) \quad a_1a_2 - a_3 > 0,$$

$$(2) \quad a_1 > 0, \quad a_2 > 0, \quad a_3 > 0,$$

which are the Routh-Hurwitz conditions for stability.

A different multiplier for equation (13) can be obtained by taking the derivative of the last two terms of that equation. The result is

$$\begin{aligned} & (a_2\ddot{x} + a_3\dot{x})(\ddot{x} + a_1\dot{x}) + (a_2\ddot{x} + a_3\dot{x})(a_2\dot{x} + a_3x) = \\ & = a_2\ddot{x}\ddot{x} + a_1a_3\ddot{x}\dot{x} + (a_2\ddot{x} + a_3\dot{x})(a_2\dot{x} + a_3x) + a_1a_2(\ddot{x})^2 + a_3\dot{x}\ddot{x} = 0. \end{aligned} \quad (21)$$

The term, $a_3\dot{x}\ddot{x}$, in equation (21) can be rewritten as

$$a_3\dot{x}\ddot{x} = a_3 \frac{d(\dot{x}\ddot{x})}{dt} - a_3(\dot{x})^2 \quad (22)$$

Combining (21) and (22), gives

$$\frac{1}{2} \frac{d}{dt} \left[a_2(\dot{x})^2 + a_1a_3(\dot{x})^2 + (a_2\dot{x} + a_3x)^2 + 2a_3\dot{x}x \right] = - (a_1a_2 - a_3)(\dot{x})^2. \quad (23)$$

Define \dot{V} as

$$\dot{V} = - (a_1a_2 - a_3)(\dot{x})^2. \quad (24)$$

Thus, V is

$$2V = (a_2\dot{x} + a_3x)^2 + a_3/a_1(a_1\dot{x} + \ddot{x})^2 + (a_2 - a_3/a_1)(\ddot{x})^2. \quad (25)$$

From (24) and (25), we see that the conditions for asymptotic stability are again the Routh-Hurwitz conditions.

To summarize, we observe that Harris obtains multipliers by integrating or differentiating certain terms in the differential equation (13). These multipliers are applied to equation (13) and the result is that certain terms are perfect

squares of the state variables; these form our \dot{V} . Other terms are time derivatives of perfect squares and thus produce our positive definite V - function. By considering the examples at the end of this section, we observe the same phenomena taking place when Harris' method is applied to nonlinear equations with one nonlinearity; except, of course, certain additional integral or derivative constraints are introduced due to the nonlinearities.

WORK OF PURI

In general, we consider an n^{th} order nonlinear, time-varying system which can be characterized by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ &\text{---} \text{---} \text{---} \text{---} \text{---} \\ \dot{x}_{n-1} &= x_n, \\ \dot{x}_n &= -f_1 x_1 - f_2 x_2 - \dots - f_n x_n, \end{aligned} \tag{26}$$

where

$$\begin{aligned} f_i &\equiv f_i(\underline{x}, t), \quad i = 1, 2, \dots, n, \\ f_i(\underline{0}, t) &= 0, \quad t \geq 0. \end{aligned} \tag{27}$$

We assume that the system in (26) possesses a unique equilibrium solution, $\underline{x} = \underline{0}$, whose stability is to be studied.

We first formulate Liapunov functions for linear, autonomous systems. The linear equation is

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n + a_{n+1} x_{n+1} = 0, \tag{28}$$

where

$$x_{k+1} = \frac{d^k x_1}{d t^k}, \quad k = 1, 2, \dots, n,$$

and a_1, \dots, a_{n+1} are constants. Denote the initial state variables at time, $t=0$, as $x_1(0), x_2(0), \dots, x_n(0)$. The first step in generating Liapunov functions is to apply a succession of multipliers to equation (28). Then through the use of integration by parts, we can eventually generate expressions which are candidates for Liapunov functions.

Multiply (28) by $2x_1$ and integrate from 0 to t :

$$2a_1 \int_0^t x_1^2 dt + 2a_2 \int_0^t x_1 x_2 dt + \dots + 2a_{n+1} \int_0^t x_1 x_{n+1} dt = 0. \quad (29)$$

By repeated integration by parts, we have

$$\int_0^t x_1 x_{2j+1} dt = \left[x_1 x_{2j} - x_2 x_{2j-1} + \dots + (-1)^{j-1} x_j x_{j+1} \right] \Big|_0^t + (-1)^j \int_0^t x_{j+1}^2 dt, \quad (30)$$

and

$$\int_0^t x_1 x_{2j} dt = \left[x_1 x_{2j-1} - x_2 x_{2j-2} + \dots + (-1)^{j-1} \frac{1}{2} x_j^2 \right] \Big|_0^t \quad (31)$$

where $j = 1, 2, \dots$. Let us adopt the notation

$$I_k = 2 \int_0^t x_k^2 dt, \quad k = 1, 2, \dots, n. \quad (32)$$

Therefore, applying (30), (31), and (32) to equation (29), there results:

$$a_1 I_1 - a_3 I_2 + a_5 I_3 + \dots = - \left[(a_2 x_1^2 + 2a_3 x_1 x_2 + 2a_4 x_1 x_3 + \dots \right. \\ \left. \dots + 2a_{n+1} x_1 x_n) - (a_4 x_2^2 + 2a_5 x_2 x_3 + \dots) + (a_6 x_3^2 + 2a_7 x_3 x_4 + \dots) + \dots \right] \Big|_0^t. \quad (33)$$

The right hand side of (33) is a quadratic form and thus (33) can be written as

$$a_1 I_1 - a_3 I_2 + a_5 I_3 + \dots + (0) I_n = - \left[\underline{x}_t \underline{\alpha}_1 \underline{x} \right] \Big|_0^t, \quad (34)$$

where

$$\underline{\alpha}_1 = \begin{array}{|c|c|c|c|c|c|} \hline a_2 & a_3 & a_4 & a_5 & a_6 & \dots \\ \hline a_3 & -a_4 & -a_5 & -a_6 & -a_7 & \dots \\ \hline a_4 & -a_5 & +a_6 & a_7 & : & \dots \\ \hline a_5 & -a_6 & a_7 & -a_8 & : & \dots \\ \hline a_6 & -a_7 & : & : & : & \dots \\ \hline : & : & : & : & : & \dots \\ \hline \end{array} \quad (35)$$

Similarly, multiplying (28) by $2x_2$, $2x_3$, ..., $2x_n$ successively, and integrating from 0 to t , the following results are obtained:

$$\begin{aligned} a_2 I_2 - a_4 I_3 + a_6 I_4 + \dots + 0 \cdot I_n &= - \left[\underline{x}_t \underline{\alpha}_2 \underline{x} \right]_0^t, \\ -a_1 I_2 + a_3 I_3 - a_5 I_4 + \dots + 0 \cdot I_n &= - \left[\underline{x}_t \underline{\alpha}_3 \underline{x} \right]_0^t, \\ \dots \dots \dots - a_{n-2} I_{n-1} + a_n I_n &= - \left[\underline{x}_t \underline{\alpha}_n \underline{x} \right]_0^t. \end{aligned} \quad (36)$$

The exact form of the last equation in (36) depends upon n being an odd or even integer. The first few $\underline{\alpha}$ - matrices are defined as

$$\underline{\alpha}_2 = \begin{array}{|c|c|c|c|c|} \hline a_1 & 0 & 0 & 0 & \dots \\ \hline 0 & a_3 & a_4 & a_5 & \dots \\ \hline 0 & a_4 & -a_5 & -a_6 & \dots \\ \hline 0 & a_5 & -a_6 & a_7 & \dots \\ \hline : & : & : & : & : \\ \hline \end{array}, \quad \underline{\alpha}_3 = \begin{array}{|c|c|c|c|c|c|} \hline 0 & a_1 & 0 & 0 & 0 & \dots \\ \hline a_1 & a_2 & 0 & a_4 & a_5 & \dots \\ \hline 0 & 0 & -a_4 & -a_5 & 0 & \dots \\ \hline 0 & a_4 & -a_5 & a_6 & 0 & \dots \\ \hline : & : & : & : & : & : \\ \hline \end{array} \quad (37)$$

where α_n can be obtained in the same fashion depending upon n being odd or even.

Now, the equations in (36) can be written in matrix form as:

$$\underline{a} \underline{I} = - \left[\underline{Q}(t) - \underline{Q}(0) \right], \quad (38)$$

where

$$\underline{a} = \begin{bmatrix} a_1 & -a_3 & a_5 & \dots & 0 \\ 0 & a_2 & -a_4 & \dots & 0 \\ 0 & -a_1 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \dots & \dots & \dots & -a_{n-2} & a_n \end{bmatrix}, \quad \underline{I} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_n \end{bmatrix}. \quad (39)$$

If the matrix \underline{a} is nonsingular, then equation (38) can be rewritten as

$$\underline{I} = - \underline{a}^{-1} \left[\underline{Q}(t) - \underline{Q}(0) \right], \quad (40)$$

where we denote \underline{a}^{-1} as $\underline{a}^{-1} = \underline{b} = [b_{ij}]$. Equation (40) is now rewritten as

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_n \end{bmatrix} = - \begin{bmatrix} \underline{x}_t(t) & \underline{S}_1 \underline{x}(t) \\ \underline{x}_t(t) & \underline{S}_2 \underline{x}(t) \\ \vdots & \vdots \\ \underline{x}_t(t) & \underline{S}_n \underline{x}(t) \end{bmatrix} + \begin{bmatrix} \underline{x}_t(0) & \underline{S}_1 \underline{x}(0) \\ \underline{x}_t(0) & \underline{S}_2 \underline{x}(0) \\ \vdots & \vdots \\ \underline{x}_t(0) & \underline{S}_n \underline{x}(0) \end{bmatrix} \quad (41)$$

where

$$\underline{S}_i = \sum_{j=1}^n b_{ij} \alpha_j, \quad i = 1, 2, \dots, n. \quad (42)$$

We now define some candidates for Liapunov functions, namely

$$V_i(t) = \underline{x}_t(t) \underline{S}_i \underline{x}(t), \quad i = 1, 2, \dots, n. \quad (43)$$

If the i th function, $V_i(t)$, is a Liapunov function, then \underline{S}_i is required to be positive definite. Referring to equations (41) and (43), we see that

$$V_i(t) - V_i(0) = - I_i(t) \quad \text{for } i = 1, 2, \dots, n. \quad (44)$$

Thus, the time derivative is

$$\dot{V}_i(t) = - \dot{I}_i(t) = - 2x_i^2; \quad i = 1, 2, \dots, n. \quad (45)$$

Therefore $\dot{V}_i(t)$ is negative semi-definite. If \underline{S}_i is positive definite, then V_i is a Liapunov function for the linear system. The conditions for asymptotic stability must then be the same as the Routh-Hurwitz conditions. This procedure for linear systems is applied to a third order case in example 13.

We now consider an n th order nonlinear system represented by

$$x_{n+1} + f_n x_n + f_{n-1} x_{n-1} + \dots + f_2 x_2 + f_1 x_1 = 0, \quad (46)$$

where the f_i 's are defined in (27). Equation (46) is rewritten as

$$x_{n+1} + a_n x_n + \dots + a_1 x_1 = - F, \quad (47)$$

where

$$F = (f_n - a_n) x_n + \dots + (f_2 - a_2) x_2 + (f_1 - a_1) x_1, \quad (48)$$

and the a_i 's are constants.

Equation (47) is multiplied by $2x_1, 2x_2, \dots, 2x_n$ successively and then integrated in the same manner as in the linear case, the result being:

$$I_i = - \left[\underline{x}_t \underline{S}_i \underline{x} \right]_0^t - 2 \int_0^t \sum_{j=1}^n \sum_{k=1}^n b_{ij} (f_k - a_k) x_j x_k dt, \quad (49)$$

or, from (32),

$$- \int_0^t \left[\underline{x}_1^2 + 2 \sum_{j=1}^n \sum_{k=1}^n b_{ij} (f_k - a_k) x_j x_k \right] dt = \left[\underline{x}_t \underline{S}_i \underline{x} \right]_0^t. \quad (50)$$

Multiplying equation (50) by a positive constant c_i and then summing over i gives

$$-\int_0^t \sum_{i=1}^n c_i (x_i^2 + 2 \sum_{j=1}^n \sum_{k=1}^n b_{ij} (f_k - a_k) x_j x_k) dt = \left[\sum_{i=1}^n c_i x_t \underline{S}_i x \right]_0^t. \quad (51)$$

Choose as a candidate for a Liapunov function

$$V(t) = \sum_{i=1}^n c_i x_t \underline{S}_i x + \quad \text{(certain terms in the integral in (51)).} \quad (52)$$

This V -function is positive definite if the \underline{S}_i are positive definite and if the integral terms are at least positive semi-definite. The c_i 's are chosen such that the time derivative is negative semi-definite. The actual procedure outlined above will be demonstrated in examples 14, 15, and 16.

In summary, we feel that the method due to Puri is much more versatile than that due to Harris, and more systematic than that due to Geiss and Reiss. Puri's method can be applied to higher order systems, but the major difficulty is still that of determining when a form is positive definite.

COMPENDIUM OF EXAMPLES

The first set of examples deals with the repeated use of integration by parts. These examples come from the papers of Geiss and Reiss. The next set of examples is obtained from Harris' thesis and deals with the generation of certain multipliers which aid in the formulation of Liapunov functions. Also, included at this point is an example by Ingwerson, [7], which explains in detail his application of the multiplier method. The last set of examples is concerned with Puri's method, which is a combination of both state variable multiplication and integration by parts.

Example 1, [2,3]

The following example illustrates the usefulness of integration by parts to modify an existing Liapunov function and its derivative.

Consider the Duffing equation for a "hard spring":

$$\ddot{x} + a \dot{x} + x + b x^3 = 0,$$

where in state variable notation, we have

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -x_1 - ax_2 - bx_1^3,$$

and $a > 0$ and $b > 0$.

As a candidate for a Liapunov function, choose

$$V_1 = 1/2 [x_1^2 + x_2^2].$$

The time derivative of V_1 along the trajectories of the system is

$$\dot{V}_1 = -ax_2^2 - bx_1^3 x_2.$$

Because of the term, $bx_1^3 x_2$, \dot{V}_1 is indefinite. As a new candidate, suppose we choose

$$\dot{V}_2 = -ax_2^2,$$

which is negative semidefinite. V_2 becomes

$$\begin{aligned} V_2 &= \int \dot{V}_2 dt = + \int -ax_2^2 dt = \int (-ax_2) x_2 dt = \\ &= \int (x_2 + x_1 + bx_1^3) x_2 dt = \frac{x_2^2}{2} + \int (x_1 + bx_1^3) \dot{x}_1 dt = \frac{x_2^2}{2} + \frac{x_1^2}{2} + b \frac{x_1^4}{4}. \end{aligned}$$

Thus, V_2 is a Liapunov function whose time derivative is negative semidefinite and $V \rightarrow \infty$ as $\| \underline{x} \| \rightarrow \infty$. Therefore, the system is globally asymptotically stable.

In order to apply LaSalle's theorem on complete stability we need a V -function whose time derivative is negative definite. In the above we have shown that complete stability exists already; but as an example, we will run through the procedure of obtaining a V_3 such that \dot{V}_3 is negative definite. Suppose that there exists a \dot{V}_3 such that

$$\dot{V}_3 = -a x_2^2 - x_1^2.$$

Thus,

$$\begin{aligned} V_3 &= \int -ax_2^2 dt - \int x_1^2 dt = V_2 + \int (-x_1) x_1 dt = \\ &= V_2 + \int (\dot{x}_2 + ax_2 + bx_1^3) x_1 dt. \end{aligned}$$

By integration by parts, the first integral becomes

$$\begin{aligned} \int x_1 \dot{x}_2 dt &= x_1 x_2 - \int \dot{x}_1 x_2 dt = x_1 x_2 - \int x_2^2 dt \\ &= x_1 x_2 + \frac{1}{a} \left(\frac{x_2^2}{2} + \frac{x_1^2}{2} + b \frac{x_1^4}{4} \right). \end{aligned}$$

The second integral becomes

$$\int ax_1 x_2 dt = \int a \dot{x}_1 x_1 dt = a \frac{x_1^2}{2};$$

thus,

$$\begin{aligned} V_3 &= \frac{x_2^2}{2} + \frac{x_1^2}{2} + b \frac{x_1^4}{4} + x_1 x_2 + \frac{1}{a} \left(\frac{x_2^2}{2} + \frac{x_1^2}{2} + b \frac{x_1^4}{4} \right) + \\ &+ a \frac{x_1^2}{2} + b \int x_1^4 dt = \frac{1}{2} (x_1 + x_2)^2 + \frac{1}{2} (a + 1/a) x_1^2 + \frac{b}{a} \left(\frac{1+a}{4} \right) x_1^4 + \\ &+ \frac{x_2^2}{2a} + \int bx_1^4 dt. \end{aligned}$$

Finally, define V_4 as

$$V_4 = V_3 - \int bx_1^4 dt$$

where

$$\dot{V}_4 = -ax_2^2 - x_1^2 - bx_1^4.$$

The V_4 - function is a Liapunov function which satisfies the conditions for complete stability required in LaSalle's theorem.

Since Duffing's equation is such a popular example, let us continue to "pump" it for information. If we replace b by $-b$, $b > 0$, our system describes a "soft spring." In this case V_2 is still a reasonable Liapunov function (in fact, it is a measure of the total energy of the undamped system, $a = 0$). V_2 also tells us

that all solutions originating in the region \mathcal{N}_l defined by

$$\mathcal{N}_l : V_2 = \frac{x_1^2}{2} + \frac{x_2^2}{2} - \frac{bx_1^4}{4} < \frac{1}{4b}$$

$$\frac{x_2^2}{x_1^2} < \frac{1}{b},$$

tend toward the origin. The origin is then asymptotically stable. This region

\mathcal{N}_l is not the complete region of asymptotic stability as shown in reference [3].

If $x_1^2 > 1/b$ and $x_1 x_2 > 0$, then the system is unstable.

Example 2, [3] Third Order Linear System

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -a_3 x_3 - a_2 x_2 - a_1 x_1,\end{aligned}$$

where a_1, a_2, a_3 , are constants. Let us select a negative semi-definite form for \dot{V} ,

$$\dot{V} = -x_3^2.$$

Then,

$$\begin{aligned}V &= - \int x_3^2 dt = -x_2 x_3 + \int x_2 \dot{x}_3 dt \\ &= -x_2 x_3 + \int x_2 (-a_3 x_3 - a_2 x_2 - a_1 x_1) dt \\ &= -x_2 x_3 - \frac{a_3}{2} x_2^2 - a_2 \int x_2^2 dt - \frac{a_1}{2} x_1^2 \\ &= -x_2 x_3 - \frac{a_3}{2} x_2^2 - \frac{a_1 x_1^2}{2} - a_2 \left[x_1 x_2 - \int x_1 x_3 dt \right].\end{aligned}$$

We integrate the last term to get

$$\begin{aligned}- \int x_1 x_3 dt &= \frac{1}{a_1} \int x_3 \left[a_3 x_3 + a_2 x_2 + \dot{x}_3 \right] dt \\ &= \frac{1}{2a_1} \left[a_2 x_2^2 + x_3^2 + 2a_3 \int x_2^2 dt \right] \\ &= \frac{1}{2a_1} \left[a_2 x_2^2 + x_3^2 - 2a_3 V \right].\end{aligned}$$

Combining the last two equations gives;

$$V = \frac{a_1}{2(a_2 a_3 - a_1)}$$

x_1	x_2	x_3
-------	-------	-------

a_1	a_2	0	x_1
a_2	$a_2^2/a_1 + a_3$	1	x_2
0	1	a_2/a_1	x_3

V is positive definite if

- 1) $a_1 > 0$,
- 2) $a_3 > 0$,
- 3) $a_2 a_3 - a_1 > 0$.

Hence the system is asymptotically stable in the large. The above inequalities are the necessary and sufficient conditions of Routh-Hurwitz.

Example 3, [2, 3], Third Order Nonlinear System

Consider the system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = - (x_1 + c x_2)^n - b x_3.$$

Choose $\dot{V}_3 = - x_3^2$ and using the integration by parts procedure:

$$\begin{aligned} V_3 &= - \int x_3^2 dt = - x_2 x_3 + \int x_2 \dot{x}_3 dt \\ &= - x_2 x_3 - \int x_2 \left[(x_1 + c x_2)^n + b x_3 \right] dt \\ &= - x_2 x_3 - b x_2^2 / 2 - \int (x_2 + c x_3) (x_1 + c x_2)^n dt + \int c x_3 (x_1 + c x_2)^n dt \\ &= - x_2 x_3 - b x_2^2 / 2 - \frac{(x_1 + c x_2)^{n+1}}{n+1} - \frac{c x_3^2}{2} - bc \int x_3^2 dt \end{aligned}$$

$$\begin{aligned}
 &= - \frac{x_2 x_3 - b x_2^2}{2} - \frac{(x_1 + c x_2)^{n+1}}{n+1} - \frac{c x_3^2}{2} + b c V_3 \\
 &= - \left[\frac{(x_1 + c x_2)^{n+1}}{n+1} + \frac{b}{2} x_2^2 + x_2 x_3 + \frac{c}{2} x_3^2 \right] + b c V_3 .
 \end{aligned}$$

Thus,

$$V_3 = \frac{1}{bc - 1} \left[\frac{(x_1 + c x_2)^{n+1}}{n+1} + \frac{b}{2} x_2^2 + x_2 x_3 + \frac{c}{2} x_3^2 \right]$$

and

$$\dot{V}_3 = - \frac{2}{x_3} .$$

Rewriting V_3 gives,

$$V_3 = \frac{1}{bc - 1} \left(\frac{(x_1 + c x_2)^{n+1}}{n+1} + \frac{1}{2} [x_2, x_3] \begin{bmatrix} b & 1 \\ 1 & c \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \right).$$

Therefore, V_3 is positive definite and the system is asymptotically stable in the large if

$$(1) \quad bc - 1 > 0,$$

$$(2) \quad b > 0,$$

$$(3) \quad n = 2k + 1, \quad k = 0, 1, 2, \dots,$$

where (1) and (2) are the Sylvester's conditions for $\begin{bmatrix} b & 1 \\ 1 & c \end{bmatrix}$. We have shown that conditions (1), (2), and (3) are sufficient. In reference [3], Geiss and Reiss show that these conditions are also necessary. This work is repeated below.

Choose $V_2 = x_2^2$. Calculate V_2 via of integration by parts procedure:

$$\begin{aligned}
 V_2 &= \int x_2^2 dt = x_1 x_2 - \int x_1 x_3 dt \\
 &= x_1 x_2 + 1/b \int \left[x_1 (x_1 + c x_2)^n + x_1 \dot{x}_3 \right] dt
 \end{aligned}$$

$$\begin{aligned}
 &= x_1 x_2 + 1/b \int (x_1 + c x_2)^{n+1} dt + \frac{1}{b} x_1 x_3 - \frac{1}{b} \int x_2 x_3 dt + \\
 &- c/b \int x_2 (x_1 + c x_2)^n dt \\
 &= x_1 x_2 + 1/b x_1 x_3 - 1/2b x_2^2 + 1/b \int (x_1 + c x_2)^{n+1} dt + \\
 &- c/b \int (x_2 + c x_3)(x_1 + c x_2)^n dt + c^2/b \int x_3 (x_1 + c x_2)^n dt \\
 &= x_1 x_2 + 1/b x_1 x_3 - 1/(2b) x_2^2 - \frac{c}{(n+1)b} (x_1 + c x_2)^{n+1} + \\
 &+ 1/b \int (x_1 + c x_2)^{n+1} dt + c^2/b \int x_3 (x_1 + c x_2)^n dt.
 \end{aligned}$$

Looking at the last integral in V_2 , we have

$$\begin{aligned}
 c^2/b \int x_3 (x_1 + c x_2)^n dt &= c^2/b \int x_3 (-\dot{x}_3 - b x_3) dt \\
 &= - \frac{c^2 x_3^2}{2b} + c^2 V_3.
 \end{aligned}$$

Therefore, we define a V_2' as

$$\begin{aligned}
 V_2' &= V_2 - 1/b \int (x_1 + c x_2)^{n+1} dt \\
 &= x_1 x_2 + 1/b x_1 x_3 - 1/2b x_2^2 - \frac{c}{b(n+1)} (x_1 + c x_2)^{n+1} - \frac{c^2 x_3^2}{2b} + c^2 V_3,
 \end{aligned}$$

and

$$\dot{V}_2' = \dot{V}_2 - 1/b (x_1 + c x_2)^{n+1} = x_2^2 - 1/b (x_1 + c x_2)^{n+1}.$$

If we now take a linear combination of V_3 and V_2' , we obtain a definite \dot{V} :

$$V = -\alpha V_3 + V_2' = \left[\frac{\alpha - c^2}{1 - bc} - \frac{c}{b} \right] \frac{(x_1 + c x_2)^{n+1}}{n+1} +$$

$$+ 1/2 [x_1, x_2, x_3]$$

0	1	1/b
1	$\left[\frac{\alpha - c^2}{1 - bc} \right] b - 1/b$	$\frac{\alpha - c^2}{1 - bc}$
1/b	$\frac{\alpha - c^2}{1 - bc}$	$\left[\frac{\alpha - c^2}{1 - bc} \right] c - \frac{c^2}{b}$

x_1
x_2
x_3

and

$$\dot{V} = \alpha x_3^2 + x_2^2 - 1/b (x_1 + cx_2)^{n+1}.$$

We see that \dot{V} is positive definite if $\alpha > 0$ and $b > 0$. If $\alpha > c^2$, $c < 0$, and $1 - (b/c)(\alpha - c^2) < bc < 1$; then V takes on positive values arbitrarily close to the origin. Thus, the origin is unstable; and $bc > 1$, $b > 0$, and $n=2k+1$ ($k = 0, 1, 2, \dots$) become necessary and sufficient conditions for stability.

Example 4, [2, 3] Third Order - Nonlinear

Consider the system

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3,$$

$$\dot{x}_3 = -F(x_2)x_3 - ax_2 - bx_1.$$

Choose $\dot{V}_3 = x_3^2$ and integrate by parts:

$$\begin{aligned} V_3 &= \int x_3^2 dt = x_2 x_3 - \int x_2 \dot{x}_3 dt \\ &= x_2 x_3 + \int [ax_2^2 + bx_1 x_2 + F(x_2)x_2 x_3] dt \\ &= x_2 x_3 + \frac{bx_1^2}{2} + \int F(x_2)x_2 dx_2 + a \int x_2^2 dt. \end{aligned}$$

Now consider the last integral:

$$\begin{aligned} a \int x_2^2 dt &= ax_1 x_2 + \frac{a}{b} \int x_3 [\dot{x}_3 + F(x_2)x_3 + ax_2] dt \\ &= ax_1 x_2 + \frac{a}{b} \frac{x_3^2}{2} + \frac{a}{b} \frac{x_2^2}{2} + \frac{a}{b} \int F(x_2)x_3^2 dt. \end{aligned}$$

Thus, V_3 becomes

$$V_3 = x_2 x_3 + \int F(x_2) x_2 dx_2 + \frac{a}{b} \frac{x_3^2}{2} + \frac{a}{b} \frac{x_2^2}{2} + \\ + ax_1 x_2 + \frac{b}{2} x_1^2 + \int \frac{a}{b} F(x_2) x_3^2 dt.$$

Define V'_3 as

$$V'_3 = V_3 - \int \frac{a}{b} F(x_2) x_3^2 dt \\ = x_2 x_3 + \int F(x_2) x_2 dx_2 + \frac{b}{2} x_1^2 + ax_1 x_2 + \\ + \frac{a}{b} \frac{x_3^2}{2} + \frac{a}{b} \frac{x_2^2}{2},$$

and

$$\dot{V}'_3 = \dot{V}_3 - \frac{a}{b} F(x_2) x_3^2 = - \frac{a}{b} \left[F(x_2) - \frac{b}{a} \right] x_3^2.$$

We can rewrite V'_3 as

$$V'_3 = \frac{1}{2b} \left[ax_2 + bx_1 \right]^2 + \frac{1}{2ab} \left[ax_3 + bx_2 \right]^2 + \\ + \int_0^{x_2} \left[F(x_2) - b/a \right] x_2 dx_2.$$

Thus, the system under discussion is globally asymptotically stable if

(1) $F(x_2) \geq c > b/a$, c being constant,

(2) $a > 0$ and $b > 0$.

Example 5, [3] Nonautonomous System

Consider the example given by

$$\ddot{x} + a(t) \dot{x} + b(t) x = 0,$$

or in state variable notation

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -ax_2 - bx_1.$$

Choose $\dot{V}_2 = x_2^2$. Integrating \dot{V}_2 gives

$$\begin{aligned} V_2 &= \int x_2^2 dt = x_1 x_2 - \int x_1 \dot{x}_2 dt \\ &= x_1 x_2 + \int a x_1 x_2 dt + \int b x_1^2 dt \\ &= x_1 x_2 + a x_1^2 - \int [a x_2 + \dot{a} x_1] x_1 dt + \int b x_1^2 dt. \end{aligned}$$

Substituting for x_1 in the term $a x_1 x_2$, we get

$$\begin{aligned} V_2 &= x_1 x_2 + a x_1^2 + \int \frac{a}{b} [a x_2^2 + x_2 \dot{x}_2] dt \\ &\quad + \int (b - \dot{a}) x_1^2 dt. \end{aligned}$$

Applying integration by parts again gives:

$$\begin{aligned} V_2 &= x_1 x_2 + a x_1^2 + \frac{a}{b} \frac{x_2^2}{2} - \int \frac{x_2^2}{2} \frac{d(a/b)}{dt} dt \\ &\quad + \int \frac{a^2}{b} x_2^2 dt + \int (b - \dot{a}) x_1^2 dt. \end{aligned}$$

Now define V_3 as

$$\begin{aligned} V_3 &= V_2 - \left[- \int \frac{x_2^2}{2} \frac{d(a/b)}{dt} dt + \int (a^2/b) x_2^2 dt + \right. \\ &\quad \left. + \int (b - \dot{a}) x_1^2 dt \right], \end{aligned}$$

which gives

$$\dot{V}_3 = \dot{V}_2 + \frac{d(a/b)}{dt} \frac{x_2^2}{2} - \frac{a^2}{b} x_2^2 - (b - \dot{a}) x_1^2,$$

or, rewriting we have

$$\begin{aligned} \dot{V}_3 &= x_2^2 + 1/2 \frac{d(a/b)}{dt} x_2^2 - \frac{a^2}{b} x_2^2 - (b - \dot{a}) x_1^2 \\ &= - \left[\frac{a^2}{b} - \frac{1}{2} \frac{d(a/b)}{dt} - 1 \right] x_2^2 - (b - \dot{a}) x_1^2. \end{aligned}$$

Thus, the sufficient conditions for asymptotic stability are

$$(1) \quad 2a^2/b - 1 > 0,$$

$$(2) \quad a^2/b - 1/2 \frac{d(a/b)}{dt} - 1 > 0,$$

$$(3) \quad \left. \begin{array}{l} a \geq k_1 > 0 \\ a < m \\ b \geq k_2 > 0 \end{array} \right\} \quad m, k_1, k_2 \text{ are constants,}$$

$$(4) \quad b - \dot{a} > 0.$$

Example 6, [4]

Consider the second order case

$$\ddot{x} + a_1 \dot{x} + a_2(x) x = 0,$$

where a_1 is a constant. Multiply the equation by \dot{x} to get

$$\dot{x} \ddot{x} + a_1 (\dot{x})^2 + a_2(x) x \dot{x} = 0.$$

Define \dot{V} as

$$\dot{V} = -a_1 (\dot{x})^2 = \dot{x} \ddot{x} + a_2(x) x \dot{x}.$$

Integrating \dot{V} gives

$$V = \int [\dot{x} \ddot{x} + a_2(x) x \dot{x}] dt = \frac{(\dot{x})^2}{2} + \int a_2(x) x \dot{x} dt.$$

Thus, for asymptotic stability, we require

$$(1) \quad a_1 > 0,$$

$$(2) \quad a_2(x) > 0, \quad x \neq 0,$$

$$(3) \quad V \rightarrow \infty \text{ as } x^2 + \dot{x}^2 \rightarrow \infty.$$

Example 7, [4]

The next second order case is a generalization of example 6,

$$\ddot{x} + a_1(x) \dot{x} + a_2(x) x = 0.$$

Multiply by \dot{x} to get

$$\dot{x} \ddot{x} + a_1(x) (\dot{x})^2 + a_2(x) x \dot{x} = 0,$$

as in the previous case, define \dot{V} as

$$\dot{V} = -a_1(x) (\dot{x})^2 = \dot{x} \ddot{x} + a_2(x) x \dot{x}.$$

Integrating \dot{V} gives

$$V = 1/2 (\dot{x})^2 + \int a_2(x) x \dot{x} dt.$$

The requirements for asymptotic stability are

- (1) $a_1(x) > 0$, $x \neq 0$,
- (2) $a_2(x) > 0$, $x \neq 0$,
- (3) $V \longrightarrow \infty$ as $x^2 + \dot{x}^2 \longrightarrow \infty$.

Example 8 [4]

We consider a special case of the equation in example 7,

$$\ddot{x} + a_1(\dot{x}) \dot{x} + a_2 x = 0.$$

Apply the differential operator, $\ddot{x} d/dt$, to the equation:

$$\ddot{x} \ddot{x} + \ddot{x} (\dot{a}_1 \dot{x} + a_1 \ddot{x}) + a_2 \ddot{x} \dot{x} = 0.$$

Let \dot{V} be defined as

$$\dot{V} = -\ddot{x} \frac{d [a_1(\dot{x}) \dot{x}]}{dt} = -a_1(\dot{x}) (\ddot{x})^2 - \dot{x} (\ddot{x})^2 \frac{da_1}{d\dot{x}}.$$

Integrating \dot{V} gives

$$\begin{aligned} V &= \int \dot{V} dt = \int (\ddot{x} \ddot{x} + a_2 \dot{x} \ddot{x}) dt \\ &= \frac{(\ddot{x})^2}{2} + a_2 \frac{(\dot{x})^2}{2} = 1/2 \left[[a_1(\dot{x}) \dot{x} + a_2 x]^2 + a_2 \dot{x}^2 \right]. \end{aligned}$$

Thus, the conditions for asymptotic stability are

- (1) $a_1(\dot{x}) + \dot{x} \frac{da_1(\dot{x})}{d\dot{x}} > 0$ if $x \neq 0$ and $\dot{x} \neq 0$,

$$(2) \quad a_2 > 0,$$

$$(3) \quad V \longrightarrow \infty \quad \text{as} \quad x^2 + \dot{x}^2 \longrightarrow \infty.$$

These conditions are more restrictive than those in examples 6 & 7 because of the presence of the derivative terms.

Example 9, 4

We now consider a third order example, namely

$$\ddot{x} + a_1 \ddot{x} + a_2(x) \dot{x} + a_3 x = 0,$$

where a_1 and a_3 are constants. To form a candidate for a Liapunov function, we first multiply the above equation by $\ddot{x} + a_1 \dot{x}$ and then integrate to get V:

$$(\ddot{x} + a_1 \dot{x}) (\ddot{x} + a_1 \ddot{x}) + (\ddot{x} + a_1 \dot{x}) a_3 x + (\ddot{x} + a_1 \dot{x}) a_2(x) \dot{x} = 0,$$

or

$$\begin{aligned} & (\ddot{x} + a_1 \dot{x}) \frac{d(\ddot{x} + a_1 \dot{x})}{dt} + (\ddot{x} + a_1 \dot{x}) \frac{a_3}{a_1} (a_1 x + \dot{x}) + \\ & + (\ddot{x} + a_1 \dot{x}) \left(-\frac{a_3}{a_1} \dot{x} + a_2(x) \dot{x} \right) = 0, \end{aligned}$$

where

$$\begin{aligned} \dot{V} &= \frac{d}{dt} \left[\frac{1}{2} (\ddot{x} + a_1 \dot{x})^2 + \frac{a_3}{2a_1} (\dot{x} + a_1 x)^2 + \frac{1}{2} (a_2(x) - \frac{a_3}{a_1}) \dot{x}^2 \right] \\ &= a_3 (\dot{x})^2 - a_1 a_2(x) (\dot{x})^2 + \frac{1}{2} (\dot{x})^2 \dot{a}_2(x) \\ &= \left[a_3 - a_1 a_2(x) + \frac{\dot{a}_2(x)}{2} \right] (\dot{x})^2. \end{aligned}$$

Integrating \dot{V} gives

$$2V = (\ddot{x} + a_1 \dot{x})^2 + \frac{a_3}{a_1} (\dot{x} + a_1 x)^2 + \frac{a_1 a_2(x) - a_3}{a_1} \dot{x}^2.$$

the required conditions for asymptotic stability are

$$(1) \quad a_1 a_2(x) - a_3 - \frac{\dot{a}_2(x)}{2} > 0, \quad x \neq 0,$$

$$(2) \quad a_1 a_2(x) - a_3 > 0, \quad x \neq 0,$$

$$(3) \quad a_1 > 0 \quad \text{and} \quad a_3 > 0.$$

Example 10, [4]

Let us consider

$$\ddot{x} + a_1(\dot{x}) \ddot{x} + a_2 \dot{x} + a_3 x = 0,$$

where a_2 and a_3 are constants. Also, let us say that $a_1(\dot{x}) = a_1 + \mu(\dot{x})$

where a_1 is constant. The multiplier in this case is $\frac{a_2 \ddot{x} + a_3 \dot{x}}{a_1 \ddot{x} + a_2 \dot{x} + a_3 x}$.

Thus, we have

$$\begin{aligned} & \ddot{x} (a_2 \ddot{x} + a_3 \dot{x}) + a_1 (a_2 \ddot{x} + a_3 \dot{x}) \ddot{x} + \\ & + \mu(\dot{x}) (a_2 \ddot{x} + a_3 \dot{x}) \ddot{x} + (a_2 \ddot{x} + a_3 \dot{x}) (a_2 \dot{x} + a_3 x) = 0, \end{aligned}$$

or

$$\begin{aligned} & \frac{d}{dt} \left[\frac{a_2}{2} (\ddot{x})^2 + \frac{a_1 a_3}{2} (\dot{x})^2 + \frac{1}{2} (a_2 \dot{x} + a_3 x)^2 \right] + \\ & + a_3 \dot{x} \ddot{x} + a_1 a_2 (\ddot{x})^2 + a_2 \mu(\dot{x}) (\ddot{x})^2 + a_3 \mu(\dot{x}) \dot{x} \ddot{x} = 0. \end{aligned}$$

Since $\dot{x} \ddot{x} = \frac{d}{dt} (\frac{1}{2} \dot{x}^2) - (\dot{x})^2$, the above equation becomes

$$\begin{aligned} & \frac{d}{dt} \left[\frac{a_2}{2} (\ddot{x})^2 + \frac{a_1 a_3}{2} (\dot{x})^2 + a_3 \dot{x} \ddot{x} + \frac{1}{2} (a_2 \dot{x} + a_3 x)^2 \right] + \\ & + a_3 \mu(\dot{x}) \dot{x} \ddot{x} = a_3 (\dot{x})^2 - a_1 a_2 (\ddot{x})^2 - a_2 \mu(\dot{x}) (\ddot{x})^2. \end{aligned}$$

Thus, define \dot{V} to be

$$\dot{V} = - [a_1 a_2 + a_2 \mathcal{H}(\dot{x}) - a_3] (\ddot{x})^2,$$

and

$$V = \frac{a_2}{2} (\ddot{x})^2 + \frac{a_1 a_3}{2} (\dot{x})^2 + a_3 \dot{x} \ddot{x} + \frac{1}{2} (a_2 \dot{x} + a_3 x)^2,$$

$$\begin{aligned} 2\dot{V} = & \left[a_2 - \frac{a_3}{a_1} \right] (\ddot{x})^2 + \frac{a_3}{a_1} (a_1 \dot{x} + \ddot{x})^2 + \\ & + (a_3 x + a_2 \dot{x})^2 + a_3 \int \mathcal{H}(\dot{x}) \dot{x} \ddot{x} dt. \end{aligned}$$

Therefore, for asymptotic stability, we require

$$(1) \quad a_2 - a_3/a_1 > 0,$$

$$(2) \quad a_3 > 0, a_1 > 0,$$

$$(3) \quad \mathcal{H}(\dot{x}) > 0 \quad \text{if} \quad \dot{x} \neq 0,$$

$$(4) \quad a_1 a_2 + a_2 \mathcal{H}(\dot{x}) - a_3 > 0 \quad \text{if} \quad \dot{x} \neq 0.$$

Example 11, [4]

The next case we consider is

$$\ddot{x} + a_1 \dot{x} + a_2 x + a_3 \mathcal{H}(x) = 0,$$

where a_1, a_2, a_3 are constants. The multiplier is $\ddot{x} + a_1 \dot{x}$.

Therefore we get:

$$\begin{aligned} & (\ddot{x} + a_1 \dot{x}) (\ddot{x} + a_1 \dot{x}) + a_2 \dot{x} \ddot{x} + a_1 a_2 (\dot{x})^2 + \\ & + a_3 x \ddot{x} + a_1 a_3 x \dot{x} + \mathcal{H}(x) \ddot{x} + a_1 \mathcal{H}(x) \dot{x} = 0, \end{aligned}$$

or

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} (\ddot{x} + a_1 \dot{x})^2 + \frac{a_2}{2} (\dot{x})^2 + \frac{a_1 a_3}{2} (x)^2 \right] + a_1 \mathcal{H}(x) \dot{x} + \\ & + a_3 \left[\frac{d}{dt} (x \dot{x}) - (\dot{x})^2 \right] + a_1 a_2 (\dot{x})^2 + \mathcal{H}(x) \ddot{x} = 0, \end{aligned}$$

or

$$\frac{1}{2} \frac{d}{dt} \left[(\ddot{x} + a_1 \dot{x})^2 + \frac{a_3}{a_1} (a_1 x + \dot{x})^2 + (a_2 - \frac{a_3}{a_1}) \dot{x}^2 \right] +$$

$$+ a_1 H(x) \dot{x} = a_3 (\dot{x})^2 - a_1 a_2 (\dot{x})^2 - H(x) \ddot{x}.$$

The V-function is then chosen as

$$2V = (\ddot{x} + a_1 \dot{x})^2 + a_3/a_1 (a_1 x + \dot{x})^2 + (a_2 - a_3/a_1) \dot{x}^2 +$$

$$+ a_1 \int H(x) dx,$$

where

$$\dot{V} = - (a_1 a_2 - a_3) (\dot{x})^2 - H(x) \ddot{x}.$$

Therefore, the conditions for asymptotic stability are:

- (1) $a_1 > 0, a_3 > 0,$
- (2) $a_1 a_2 - a_3 > 0,$
- (3) $\int H(x) dx > 0,$
- (4) $H(x) \ddot{x} > 0$ if $x \neq 0.$

Example 12, [4]

Consider:

$$\ddot{x} + a_1 \ddot{x} + a_2 (\dot{x}) \dot{x} + a_3 x = 0,$$

where a_1 and a_3 are constants and a_2 is a function of \dot{x} , while in example

9, a_2 was a function of X . The multiplier is $\ddot{x} + a_1 \dot{x}$. Thus

$$(\ddot{x} + a_1 \ddot{x}) (\ddot{x} + a_1 \dot{x}) + a_2 (\dot{x}) \dot{x} \ddot{x} + a_1 a_3 x \dot{x} +$$

$$+ a_3 x \ddot{x} + a_1 a_2 (\dot{x}) (\dot{x})^2 = 0,$$

or

$$1/2 \frac{d}{dt} \left[(a_1 \dot{x} + \ddot{x})^2 + a_1 a_3 x^2 \right] + a_3 \left[\frac{d(x\dot{x})}{dt} - (\dot{x})^2 \right] +$$

$$+ a_2 (\dot{x}) \dot{x} \ddot{x} + a_1 a_2 (\dot{x}) (\dot{x})^2 = 0,$$

or

$$\frac{1}{2} \frac{d}{dt} \left[(a_1 \dot{x} + \ddot{x})^2 + a_1 a_3 (x)^2 + 2a_3 x \dot{x} + a_3/a_1 (\dot{x})^2 \right] + \left[a_2(\dot{x}) - a_3/a_1 \right] \dot{x} \ddot{x} = - (a_1 a_2(\dot{x}) - a_3) (\dot{x})^2.$$

Let \dot{V} be given by

$$\dot{V} = - (a_1 a_2(\dot{x}) - a_3) (\dot{x})^2,$$

where

$$2V = (a_1 \dot{x} + \ddot{x})^2 + a_3/a_1 (a_1 x + \dot{x})^2 + \int \left[a_2(\dot{x}) - \frac{a_3}{a_1} \right] \dot{x} \ddot{x} dt.$$

The conditions for asymptotic stability are

- (1) $a_1 a_2(\dot{x}) - a_3 > 0$ if $\dot{x} \neq 0$,
- (2) $a_1 > 0, a_3 > 0$,
- (3) $V \longrightarrow \infty$ as $x^2 + \dot{x}^2 + \ddot{x}^2 \longrightarrow \infty$.

Example 13, [7]

Ingwerson considers a third order system from control theory which has a nonlinear gain and a derivative feedback element. The problem reduces to the consideration of the following equation of motion:

$$\ddot{x} + b_1 \ddot{x} + (b_2 + b_3 c_2) \dot{x} + b_3 x + b_4 x^3 + 3b_4 c_2 x^2 \dot{x} + 3b_4 c_2^2 x \dot{x}^2 + b_4 c_2^3 \dot{x}^3 = 0,$$

where b_1, b_2, b_3, c_2 are constants. We wish to study the stability of the equilibrium solution. Ingwerson used a "multiplier" or "integrating factor" approach to generate the Liapunov function used in the analysis. We present this method in the following paragraphs.

Let the equation of motion be divided into linear and nonlinear parts:

$$[L] \equiv \ddot{x} + b_1 \ddot{x} + (b_2 + b_3 c_2) \dot{x} + b_3 x,$$

and

$$[N] \equiv b_4 (x^3 + 3c_2 x^2 \dot{x} + 3c_2^2 x \dot{x}^2 + c_2^3 \dot{x}^3) .$$

First consider the equation $[L] = 0$. The first and third terms of $[L]$ can be integrated if we multiply by \ddot{x} . Therefore,

$$\ddot{x} [L] \equiv \frac{d}{dt} \left[\frac{\ddot{x}^2}{2} + (b_2 + b_3 c_2) \frac{\dot{x}^2}{2} \right] + b_1 \ddot{x}^2 + b_3 x \ddot{x} = 0$$

The term in the square brackets is not definite and thus cannot be used as a Liapunov function. The second and fourth terms of $[L]$ can be integrated if \dot{x} is used as a multiplier:

$$\dot{x} [L] \equiv \frac{d}{dt} \left[\frac{b_1 \dot{x}^2}{2} + \frac{b_3 x^2}{2} \right] + \dot{x} \ddot{x} + (b_2 + b_3 c_2) \dot{x}^2 = 0.$$

The bracketed term in this expression is also semi-definite. Thus, we try a linear combination of \dot{x} and \ddot{x} : namely,

$$(b_1 \dot{x} + \ddot{x}) [L] \equiv \frac{d}{dt} \left[\frac{\ddot{x}^2}{2} + (b_1^2 + b_2 + b_3 c_2) \frac{\dot{x}^2}{2} + b_1 b_3 \frac{x^2}{2} \right] + b_1 \dot{x} \ddot{x} + b_3 x \ddot{x} + b_1 \ddot{x}^2 + b_1 (b_2 + b_3 c_2) \dot{x}^2 = 0.$$

Since

$$\dot{x} \ddot{x} + \ddot{x}^2 = \frac{d(\ddot{x}\dot{x})}{dt} \quad \text{and} \quad \frac{d(x\dot{x})}{dt} = x \ddot{x} + (\dot{x})^2 ,$$

then the above can be written as

$$(b_1 \dot{x} + \ddot{x}) [L] = \frac{d}{dt} \left[\frac{\ddot{x}^2}{2} + b_1 \dot{x} \ddot{x} + (b_1^2 + b_2 + b_3 c_2) \frac{\dot{x}^2}{2} + b_3 x \dot{x} + b_1 b_3 \frac{x^2}{2} \right] + (b_1 b_2 + b_1 b_3 c_2 - b_3) \dot{x}^2 = 0.$$

Thus, for the linear equation, $[L] = 0$, we choose the above bracketed term as a candidate for a Liapunov function. Its time derivative follows automatically from the above equation. Therefore V_L and \dot{V}_L are

$$V_L = \ddot{x}^2/2 + b_1 \dot{x} \ddot{x} + (b_1^2 + b_2 + b_3 c_2) \dot{x}^2/2 + b_3 x \dot{x} + b_1 b_3 x^2/2 ,$$

and

$$\dot{V}_L = - (b_1 b_2 + b_1 b_3 c_2 - b_3) \dot{x}^2.$$

Consequently, the following conditions must be fulfilled if the linear part is asymptotically stable:

$$(1) b_1 b_2 + b_1 b_3 c_2 - b_3 > 0,$$

$$(2) b_1 b_3 > 0,$$

$$(3) b_1 b_3 (b_2 + b_3 c_2) - b_3^2 > 0.$$

Ingwenson now applies the same integrating factor to the nonlinear part,

[N], of the equation of motion. Thus, after applying the multiplier,

$b_1 \dot{x} + \ddot{x}$ to [N] and then integrating by parts, the result is

$$(b_1 \dot{x} + \ddot{x}) [N] \equiv \frac{d}{dt} \left[b_4 c_2^2 x \dot{x}^3 + \frac{b_4 c_2^3}{4} \dot{x}^4 + \frac{3}{2} c_2 b_4 x^2 \dot{x}^2 + b_4 x^3 \dot{x} + b_1 b_4 x^4 / 4 \right] + b_4 (b_1 c_2 - 1) (3x^2 + 3c_2 \dot{x} x + c_2^2 \dot{x}^2) \dot{x}^2.$$

Let the bracketed term be V_n . Computing the partial derivatives,

$$\frac{\partial V_n}{\partial x} \text{ and } \frac{\partial V_n}{\partial \dot{x}},$$

we see that V_n is monotonically increasing in x and \dot{x} and is zero only at $x = \dot{x} = 0$. Thus, V_n is semi-definite in x , \dot{x} , and \ddot{x} . The time derivative of V_n is

$$\dot{V}_n = - b_4 (b_1 c_2 - 1) (3x^2 + 3c_2 x \dot{x} + c_2^2 \dot{x}^2) \dot{x}^2.$$

Now for the original equation of motion, [L] + [N] = 0, we choose as a Liapunov function

$$V = V_L + V_n,$$

or

$$V = b_1 b_3 \frac{\dot{x}^2}{2} + b_3 x \dot{x} + (b_1^2 + b_2 + c_2 b_3) \frac{\ddot{x}^2}{2} + b_1 \dot{x} \ddot{x} + \frac{\ddot{x}^2}{2} + b_1 b_4 \frac{x^4}{4} + b_4 x^3 \dot{x} + \frac{3}{2} c_2 b_4 x^2 \dot{x}^2 + b_4 c_2^2 x \dot{x}^3 + b_4 c_2^3 \frac{\dot{x}^4}{4},$$

where the time derivative is

$$\dot{V} = - \left(b_1 b_2 + (b_1 c_2 - 1) \left[b_3 + b_4 (3x^2 + 3c_2 x \dot{x} + c_2^2 \dot{x}^2) \right] \right) \dot{x}^2.$$

\dot{V} is non-positive for:

$$b_1 b_2 + (b_1 c_2 - 1) \left[b_3 + b_4 (3x^2 + 3c_2 x \dot{x} + c_2^2 \dot{x}^2) \right] > 0.$$

Thus, the system is asymptotically stable if the linear part is asymptotically stable and the above inequality is satisfied.

Example 14 [5] Third Order Linear Case

Consider a linear, time-invariant, third order system

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4 = 0,$$

where $x_i = d^i x / dt^i$. We now apply the various multipliers, $2x_1$, $2x_2$, and $2x_3$ to the equation and integrate:

$$\int_0^t 2x_1 (a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) dt = 0,$$

or

$$a_1 I_1 - a_3 I_2 = - \left[a_2 x_1^2 + 2a_3 x_1 x_2 + 2a_4 x_1 x_3 - a_4 x_2^2 \right]_0^t,$$

where

$$I_k = 2 \int_0^t x_k^2 dt, \text{ and } \underline{\alpha}_1 =$$

a_2	a_3	a_4
a_3	$-a_4$	0
a_4	0	0

Thus, we have

$$a_1 I_1 - a_3 I_2 = - \left[\underline{x}_T \underline{\alpha}_1 \underline{x} \right]_0^t$$

where $\underline{x}_T = \begin{bmatrix} x_1, x_2, x_3 \end{bmatrix}$. The other analogous expressions are:

$$\int_0^t 2x_2 (a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) dt = 0,$$

or

$$a_2 I_2 - a_4 I_3 = - \left[\underline{x}_T \underline{\alpha}_2 \underline{x} \right]_0^t,$$

a_1	0	0
0	a_3	a_4
0	a_4	0

and

$$\int_0^t 2x_3 (a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4) dt = 0,$$

or

$$-a_1 I_2 + a_3 I_3 = - \left[\underline{x}_T \underline{\alpha}_3 \underline{x} \right]_0^t,$$

0	a_1	0
a_1	a_2	0
0	0	a_4

The above equations can be written as a matrix equation

a_1	$-a_3$	0
0	a_2	$-a_4$
0	$-a_1$	a_3

I_1
I_2
I_3

$$= \underline{Q}(0) - \underline{Q}(t),$$

where $Q_T = \begin{bmatrix} x_T \otimes_1 x & x_T \otimes_2 x & x_T \otimes_3 x \end{bmatrix}$.

Solving for $[I_1, I_2, I_3]$ yields

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \frac{1}{a_1(a_2a_3 - a_1a_4)} \begin{bmatrix} a_2a_3 - a_1a_4 & a_3^2 & a_3a_4 \\ 0 & a_1a_3 & a_1a_4 \\ 0 & a_1^2 & a_1a_2 \end{bmatrix} (Q(0) - Q(t))$$

Let $\Delta = a_2a_3 - a_1a_4$ and choose V as follows:

$$\begin{aligned} V &= x_t \underline{s}_3 x \\ &= x_t \begin{bmatrix} b_{31} \otimes_1 + b_{32} \otimes_2 + b_{33} \otimes_3 \end{bmatrix} x \\ &= x_t \frac{1}{a_1 \Delta} \begin{bmatrix} a_1^3 & a_1^2 a_2 & 0 \\ a_1^2 a_2 & a_1^2 a_3 + a_1^2 a_2 & a_1^2 a_4 \\ 0 & a_1^2 a_4 & a_1 a_2 a_4 \end{bmatrix} x, \end{aligned}$$

where

$$\dot{V} = -2x_3^2.$$

V is positive definite if \underline{s}_3 is positive definite; thus, the required conditions are

$$(1) \quad a_1 > 0, \quad a_3 > 0, \quad a_4 > 0, \quad a_2 > 0,$$

$$(2) \quad a_2 a_3 - a_1 a_4 > 0,$$

which are the Routh-Hurwitz conditions for asymptotic stability.

Example 15, [5]

We now consider a third order nonlinear differential equation:

$$\ddot{x} + g_3 (\ddot{x}) \ddot{x} + a_2 \dot{x} + a_1 x = 0,$$

or in index notation,

$$a_1 x_1 + a_2 x_2 + g_3(x_3) x_3 + x_4 = 0,$$

where $\dot{x}_i = dx_i/dt$. Also, we assume $g_3(x_3)$ has the form

$f = (g_3 - a_3)x_3$, where a_3 is a constant. From equation (49) and example 14, we have:

$$\begin{aligned} I_1 &= - \left[\underline{x}_T \underline{s}_1 \underline{x} \right]_0^t - \frac{2}{a_1} \int_0^t f(x_3) x_1 x_3 dt - \frac{2a_3}{a_1 \Delta} \int_0^t f(x_3) x_3 x_2^2 dt + \\ &\quad - \frac{2a_3}{\Delta} \int_0^t f(x_3) x_3^2 dt. \\ I_2 &= - \left[\underline{x}_T \underline{s}_2 \underline{x} \right]_0^t - \frac{2a_3}{\Delta} \int_0^t f(x_3) x_2 x_3 dt - \frac{2}{\Delta} \int_0^t f(x_3) x_3^2 dt, \\ I_3 &= - \left[\underline{x}_T \underline{s}_3 \underline{x} \right]_0^t - \frac{2a_1}{\Delta} \int_0^t f(x_3) x_2 x_3 dt - \frac{2a_2}{\Delta} \int_0^t f(x_3) x_3^2 dt, \end{aligned}$$

where \underline{s}_i and Δ are defined in example 14. I_2 and I_3 are less complicated than I_1 ; thus, we will let $C_1 = C_3 = 0$ and $C_2 = 1$ in equation (52). Hence, we choose as a Liapunov function

$$\begin{aligned} V(t) &= \underline{x}_T \underline{s}_2 \underline{x} + \frac{2a_3}{\Delta} \int_0^t f(x_3) x_2 x_3 dt, \\ &= \underline{x}_T \underline{s}_2 \underline{x} + \frac{2a_3}{\Delta} \int_0^{x_2} f(x_3) x_2 dx_2, \end{aligned}$$

where $\underline{x} = \underline{0}$ is the equilibrium solution. The time derivative of V is obtained from the equation for I_2 and from equation (45):

$$\dot{V}(t) = -2x_2^2 - (2/\Delta) f(x_3) x_3^2.$$

The conditions for asymptotic stability which are derived from V and \dot{V} are

- (1) $a_1 > 0, a_2 > 0, a_3 > 0,$
- (2) $a_2 a_3 - a_1 > 0,$
- (3) $f(x_3) = g_3(x_3) - a_3 \geq 0.$

Example 16, [5]

Consider a nonlinear differential equation with three nonlinearities:

$$\ddot{x} + g_3(\ddot{x}) \ddot{x} + g_2(\dot{x}) \dot{x} + g_1(x) x = 0.$$

Rewriting in index notation, we have

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + x_4 = (a_1 - g_1) x_1 + (a_2 - g_2) x_2 + (a_3 - g_3) x_3,$$

where a_1 , a_2 , and a_3 are constants. From equation (49) and example 14, we

write I_2 as

$$\begin{aligned} I_2 = & - \left[\underline{x}_T \underline{s}_2 \underline{x} \right]_0^t - \frac{2a_1}{\Delta} \int_0^t (g_1 - a_1) x_1 x_2 dt + \\ & - \frac{2a_2}{\Delta} \int_0^t (g_2 - a_2) x_2^2 dt - \frac{2a_3}{\Delta} \int_0^t (g_3 - a_3) x_2 x_3 dt + \\ & - \frac{2}{\Delta} \int_0^t (g_3 - a_3) x_3^2 dt - \frac{2}{\Delta} \int_0^t (g_2 - a_2) x_2 x_3 dt + \\ & - \frac{2}{\Delta} \int_0^t (g_1 - a_1) x_1 x_3 dt. \end{aligned}$$

The last integral can be written as

$$\int_0^t (g_1 - a_1) x_1 x_3 dt = (g_1 - a_1) x_1 x_2 - \int_0^t \left[\frac{dg_1}{dx_1} x_1 + g_1 - a_1 \right] x_2^2 dt,$$

where $\underline{x} = \underline{0}$ is taken as the equilibrium solution. We now choose $V(t)$ as

$$V(t) = \underline{x}_T \underline{s}_2 \underline{x} + \frac{2}{\Delta} (g_1 - a_1) x_1 x_2 + \frac{2a_3}{\Delta} \int_0^{x_1} (g_1 - a_1) x_1 dx_1 +$$

$$+ \frac{2}{\Delta} \int_0^{x_2} \left[(g_2 - a_2) + a_3 (g_3 - a_3) \right] x_2 dx_2;$$

or from the equation for I_2 , we have

$$V(t) = -I_2 - \frac{2a_3}{\Delta} \int_0^t (g_2 - a_2) x_2^2 dt +$$

$$- \frac{2}{\Delta} \int_0^t (g_3 - a_3) x_3^2 dt +$$

$$- \frac{2}{\Delta} \int_0^t \left(\frac{dg_1}{dx_1} x_1 + g_1 - a_1 \right) x_2^2 dt.$$

Thus, the time derivative of $V(t)$ is

$$\dot{V}(t) = -2x_2^2 - \frac{2a_3}{\Delta} (g_2 - a_2) x_2^2 - \frac{2}{\Delta} (g_3 - a_3) x_3^2 +$$

$$+ \frac{2}{\Delta} \left[\frac{dg_1}{dx_1} x_1 + g_1 - a_1 \right] x_2^2,$$

$$= - \frac{2}{\Delta} \left[a_3 g_2 - g_1 - \frac{dg_1}{dx_1} x_1 \right] - \frac{2}{\Delta} (g_3 - a_3) x_3^2.$$

The conditions for asymptotic stability are

- (1) $a_1, a_2, a_3 > 0,$
- (2) $a_2 a_3 - a_1 > 0,$
- (3) $g_1 - a_1, g_2 - a_2, g_3 - a_3 > 0,$
- (4) $a_3 g_2 - g_1 - \frac{dg_1}{dx_1} x_1 \geq 0.$

It should be noted that the first two terms in $V(t)$, namely

$$\underline{x}_t \underline{s}_2 \underline{x} + \frac{2}{\Delta} (g_1 - a_1) x_1 x_2$$

can be written as

$$\underline{x}_t \underline{s}_2^* \underline{x} \text{ where } \underline{s}_2^* = \frac{1}{\Delta}$$

$a_1 a_3$	g_1	0
g_1	$a_3^2 + a_2$	a_3
0	a_3	1

Stability also requires that \underline{s}_2^* be positive definite.

Example 17, [5]

Now we consider the same nonlinear system as in example 15, except for the addition of a time-varying forcing function, $p(t)$. The equation is

$$\ddot{x} + g_3 (\ddot{x}) \ddot{x} + g_2 (\dot{x}) \dot{x} + g_1(x) x = p(t),$$

or in index notation

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + x_4 = (a_1 - g_1) x_1 + (a_2 - g_2) x_2 + (a_3 - g_3) x_3 + p(t),$$

where a_1, a_2, a_3 are constants. Again, let us consider the formulation for I_2 :

$$\begin{aligned} I_2 = & - \left[\underline{x}_t \underline{s}_2 \underline{x} \right]_0^t - \frac{2a_3}{\Delta} \int_0^t (g_1 - a_1) x_1 x_2 dt + \\ & - \frac{2a_3}{\Delta} \int_0^t (g_2 - a_2) \left[x_2 - \frac{p(t)}{g_2 - a_2} \right]^2 dt + \\ & - \frac{2}{\Delta} \int_0^t (g_3 - a_3) \left[x_3 - \frac{p(t)}{g_3 - a_3} \right]^2 dt + \\ & - \frac{2}{\Delta} \int_0^t \left[(g_2 - a_2) + a_3 (g_3 - a_3) \right] x_2 x_3 dt + \\ & - \frac{2}{\Delta} \int_0^t (g_1 - a_1) x_1 x_3 dt + \int_0^t \left[\frac{a_3}{\Delta (g_2 - a_2)} + \right. \\ & \left. + \frac{1}{\Delta (g_3 - a_3)} \right] p^2(t) dt. \end{aligned}$$

If there exists certain positive constants K_1 , K_2 and K_3 such that

$$(1) \quad g_2 - a_2 \geq k_2$$

$$(2) \quad g_3 - a_3 \geq k_3$$

$$(3) \quad 1/\Delta \left[1/K_3 + a_3/k_2 \right] \int_0^t p^2(t) dt \leq k_1$$

then the Liapunov function can be chosen as

$$\begin{aligned} V(t) = & \left[x_T \quad s_2^* \quad x \right] + \frac{2a_3}{\Delta} \int_0^{x_1} (g_1 - a_1) x_1 dx_1 + \\ & + \frac{2}{\Delta} \int_0^{x_2} \left[(g_2 - a_2) + a_3 (g_3 - a_3) \right] x_2 dx_2 + \\ & + 2 \left[k_1 - \frac{1}{\Delta} \int_0^t \left[\frac{a_3}{g_2 - a_2} + \frac{1}{g_3 - a_3} \right] p^2(t) dt \right], \end{aligned}$$

and

$$\begin{aligned} \dot{V}(t) = & - \frac{2x_2^2}{\Delta} \left[a_2 a_3 - g_1 - \frac{dg_1}{dx_1} x_1 \right] - \frac{2a_3}{\Delta} (g_2 - a_2) \text{ times} \\ & \left[x - \frac{p(t)}{g_2 - a_2} \right]^2 - \frac{2}{\Delta} (g_3 - a_3) \left(x_3 - \frac{p(t)}{g_3 - a_3} \right)^2. \end{aligned}$$

\dot{V} was derived from the equation for I_2 . (The notation in this example is the same as in the previous examples.) For asymptotic stability we require:

$$(1) \quad a_1, a_2, a_3 > 0,$$

$$(2) \quad g_1 - a_1, g_2 - a_2, g_3 - a_3 \geq k_3 > 0 \quad \text{and} \quad g_2 - a_2 \geq k_2 > 0,$$

$$(3) \quad (1/\Delta) (a_3/k_2 + 1/k_3) \int_0^t p^2(t) dt \leq k_1,$$

$$(4) \quad a_2 a_3 - g_1 - (dg_1/dx_1) x_1 \geq 0,$$

$$(5) \quad a_2 a_3 - a_1 > 0.$$

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SECTION FIVE

THE VARIABLE GRADIENT METHOD

Prepared by: R. L. Drake
T. Gifford

THE VARIABLE GRADIENT METHOD

SUMMARY

In this section the variable gradient technique of generating Liapunov functions is discussed. Modifications of this method are also considered. A large compendium of second, third and fourth order examples is given at the end of the section.

INTRODUCTION

In 1962, Schultz and Gibson published the results of Schultz's thesis investigation, [1]*. The idea behind their procedure is based on the results of Massera's work concerning the existence of Liapunov functions for certain asymptotically stable systems. If a V-function exists, then we assume that its gradient, ∇V , also exists. If ∇V is known, then V and \dot{V} can be determined. Therefore, their procedure is to choose the form of the variable gradient, ∇V , such that the line integral

$$V(\underline{x}) = \int_0^{\underline{x}} \nabla V \cdot d \underline{x}$$

is independent of the path of integration and such that V and \dot{V} satisfy the proper conditions of definiteness and closedness. This method of generating Liapunov functions essentially begins in the middle of Ingwerson's procedure, which is described in section [7] of this report. The systems which will be analyzed by this method are thn order, nonlinear, autonomous systems described by:

$$\begin{aligned} \dot{\underline{x}} &= \underline{f}(\underline{x}), \\ \underline{f}(0) &= 0. \end{aligned} \tag{1}$$

The discussion of the variable gradient method is presented in many references; such as, references [1] through [7].

In reference [8], Puri begins his analysis by choosing a simpler form for

* The numbers in the square brackets, [], refer to the references at the end of the section.

∇V than that of Schultz and Gibson. For certain problems this choice for ∇V gives results with more ease than the original method. Also, Puri makes use of matrix algebra to a greater extent than Schultz and Gibson. The disadvantage of Puri's simplification is that it is not as versatile as the original method.

Ku and Puri, [9] and [10], modified the variable gradient technique. In their analysis of equation (1), they assumed a "generalized" quadratic form for V ; namely,

$$V = \underline{x}_T \underline{S} \underline{x} . \quad (2)$$

The \underline{S} matrix is symmetric and the elements are functions of the state variables. The elements of \underline{S} are chosen such that the \underline{B} matrix in

$$\dot{V} = (\nabla V)_T \dot{\underline{x}} = \underline{x}_T \underline{B} \dot{\underline{x}}, \quad (3)$$

is a form which is very nearly that of the matrices used by Schultz and Gibson in the variable gradient method. Combining equations (1) and (3), Ku and Puri form the following equation for \dot{V} ,

$$\dot{V} = \underline{x}_T \underline{T} \underline{x} . \quad (4)$$

Therefore, the equilibrium solution $\underline{x} = \underline{0}$, is asymptotically stable if \underline{T} is negative semi-definite and \underline{S} is positive definite, along with certain closedness properties being satisfied. The \underline{T} matrix in equation (4) is called the Liapunov stability matrix. The authors describe two procedures to determine the stability conditions of the system in (1) from the properties imposed upon the \underline{S} and \underline{T} matrices. A detailed discussion of these procedures is found in Mekel's thesis, reference [11].

In reference [12], Puri combines some of the concepts of references [8] and [9] to arrive at a more systematic approach for the generation of Liapunov functions.

In the examples at the end of the section, a linear, time-varying system is analyzed by this method.

The work of Ho, Goldwyn and Narendra, [3], [4], and [5], considers a procedure which is similar to the variable gradient method. This new method generates Liapunov functions for nonlinear systems of low order by generalizing the concept of a common Liapunov function for a linear, variable-parameter system. The common Liapunov function is a V-function which is a quadratic form, $\underline{x}_T \underline{P} \underline{x}$, where the choice of \underline{P} depends upon the intersection of certain sets of matrices. This choice of V is more useful than choosing an arbitrary positive definite quadratic form; and the problem of determining the intersection of sets is more tractable than some of the other methods of finding V-functions.

At the end of this section is a sizable compendium of examples, due in a large part to the Ph. D. thesis of Mekel, [1].

WORK OF SCHULTZ AND GIBSON, [1] THROUGH [7]

We want to analyze the stability properties of the equilibrium solution, $\underline{x} = \underline{0}$, of equation (1). The variable gradient technique of generating Liapunov functions begins with the choice of a certain form for \underline{V} , namely,

$$\underline{V} = \underline{B} \underline{x} =$$

a_{11}	a_{12}	$\cdot \cdot \cdot$	$a_{1,n-1}$	a_{1n}
a_{21}	a_{22}	$\cdot \cdot \cdot$	$a_{2,n-1}$	a_{2n}
\cdot	\cdot	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot
\cdot	\cdot	\cdot	\cdot	\cdot
$a_{n-1,1}$	$a_{n-1,2}$	$\cdot \cdot \cdot$	$a_{n-1,n-1}$	$a_{n-1,n}$
a_{n1}	a_{n2}	$\cdot \cdot \cdot$	$a_{n,n-1}$	a_{nn}

x_1
x_2
\cdot
\cdot
\cdot
x_{n-1}
x_n

(5)

The elements in \underline{B} have the following form:

$$\begin{aligned} a_{ij} &= b_{ij} + c_{ij}(x_1, x_2, \dots, x_{n-1}), \\ (i &\neq j) \end{aligned} \quad (6)$$

$$b_{ij} = \text{constants, for } i, j = 1, 2, \dots, n,$$

and

$$\begin{aligned} a_{ii} &= b_{ii} + c_{ii}(x_i), \\ \text{for } i &= j = 1, 2, \dots, n-1, \end{aligned} \quad (7)$$

$$a_{nn} = 2.$$

The state variable, x_n , is treated as a special case because x_n usually appears linearly in the equations of control theory. Thus, if $a_{nn} = 2$, we have x_n appearing in the V-function as x_n^2 , with a unit coefficient. Once ∇V 's form is chosen, then \dot{V} can be obtained from equations (1), (3), and (5). Also, from equation (5) and the line integral

$$V = \int_{\underline{o}}^{\underline{x}} \nabla V \cdot d\underline{x}, \quad (8)$$

we can determine V if the 'independence of path' restriction is imposed. From calculus, we know that the integral in (8) is independent of path if ∇V satisfies the $n(n-1)/2$ curl equations:

$$\frac{\partial (\nabla V)_i}{\partial x_j} = \frac{\partial (\nabla V)_j}{\partial x_i} \quad (9)$$

where $(\nabla V)_i$ is the i^{th} component of ∇V . The object of this procedure is to choose the constants, b_{ij} , and the functions, c_{ij} , in equations (5), (6), and (7) such that V is a Liapunov function with the desired properties.

The following is a stepwise procedure to be followed in the application of the variable gradient technique.

- 1) Choose ∇V as in equations (5), (6), and (7).
- 2) Compute $\dot{V} = (\nabla V)_T \underline{f}(\underline{x})$. By choosing the b_{ij} 's and c_{ij} 's to be certain quantities, make \dot{V} negative semi-definite. (This is basically a trial-and-error procedure).
- 3) Apply the $n(n-1)/2$ curl equations, (9), to ∇V to determine the remaining unknowns.
- 4) Recheck \dot{V} to see if it is still negative semi-definite.
- 5) Determine V by the line integral in (8) and find the region of asymptotic stability. Since the line integral is independent of path, the most convenient method of evaluation is

$$V = \int_0^{x_1} \frac{\partial V}{\partial x_1} dx_1 + \int_0^{x_2} \frac{\partial V}{\partial x_2} dx_2 + \int_0^{x_3} \frac{\partial V}{\partial x_3} dx_3 + \dots + \int_0^{x_n} \frac{\partial V}{\partial x_n} dx_n.$$

($x_2 = \dots = x_n = 0$) ($x_3 (= \dots = x_n = 0)$) ($x_4 = \dots = x_n = 0$)

- 6) Check the closedness of V ; that is, we must show that

$$\lim_{\|\underline{x}\| \rightarrow \infty} V(\underline{x}) = \infty.$$

PURI'S WORK [8]

We again consider equation (1). Puri assumes that the gradient of a V_1 -function has the form given by

$$\nabla V_1 = \underline{B} \underline{x},$$

where \underline{B} is symmetric and $b_{nn} = 1$. Then, \dot{V}_1 is formed:

(10)

$$\dot{V}_1 = (\nabla V_1)_T \underline{\dot{x}} = \underline{x}_T \underline{B}_T \underline{A} \underline{x} = \underline{x}_T \underline{T} \underline{x},$$

where $\underline{f}(\underline{x})$ in equation (1) is written as $\underline{A}(\underline{x})\underline{x}$, and matrix $\underline{T} = \underline{B}_T \underline{A}$. Since \dot{V}_1

is expressed in a "quadratic form", then the matrix function \underline{T} can be written in an equivalent triangular form:

$$\underline{T}_e = \begin{array}{c|ccccc} & t_{11} & t_{12} & t_{13} & \cdot & \cdot & \cdot & t_{1,n-1} & t_{1,n} \\ \hline 0 & & t_{22} & t_{23} & \cdot & \cdot & \cdot & t_{2,n-1} & t_{2,n} \\ 0 & 0 & & t_{33} & \cdot & \cdot & \cdot & t_{3,n-1} & t_{3,n} \\ \hline \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline 0 & 0 & 0 & \cdot & \cdot & \cdot & & t_{n-1,n-1} & t_{n-1,n} \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & & 0 & t_{nn} \end{array} \quad (12)$$

There are $\left[\frac{n(n+1)}{2} - 1 \right]$ unknown elements in matrix \underline{B} . Thus, we need the same number of independent relationships involving the elements of \underline{B} . If all the elements in \underline{T}_e , except one of the diagonal elements, are set equal to zero, we have the required number of relationships involving the elements of \underline{B} ; and by our choice of which element of \underline{T}_e is nonzero, matrix \underline{T}_e can be made semi-definite. We next check the elements of \underline{B} to see if the curl equations, (9), are satisfied. It is at this point Puri's modification may cause trouble. In order to check the curl equations, the symmetry of \underline{B} may have to be altered and a new V-function formed. Then, the next step is to recheck the semi-definiteness of \underline{T}_e . If this "checks out", we integrate ∇V as in the previous method to obtain the final Liapunov function, V. In the examples, Puri's modification will be amply illustrated.

THE WORK OF KU AND PURI, [9] and [10]

Ku and Puri consider the stability of the equilibrium solution of a n^{th} order, autonomous system described by

$$x^{(n)} + a_n x^{(n-1)} + \dots + a_1 x = 0, \quad (13)$$

where the a 's are functions of the variables $x, x^{(1)}, \dots, x^{(n-1)}$. The symbol $x^{(i)}$ is defined as the i -th derivative of x with respect to t . In the usual

matrix formulation, equation (13) is written as

$$\dot{\underline{x}} = \underline{A}(\underline{x})\underline{x} = \underline{A} \underline{x}, \quad (14)$$

where $x_1 = x$, $x_2 = x^{(1)}$, $x_3 = x^{(2)}$, ..., $x_n = x^{(n-1)}$, and

$$\underline{A} = \begin{array}{c|ccccccc} \begin{array}{c} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -a_1 \end{array} & \begin{array}{c} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -a_2 \end{array} & \begin{array}{c} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -a_3 \end{array} & \begin{array}{c} 0 \\ 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -a_4 \end{array} & \begin{array}{c} \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -a_{n-1} \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ -a_n \end{array} \\ \hline \end{array} \quad (15)$$

It is assumed that the coordinate system is such that the equilibrium solution is $\underline{x} = \underline{0}$.

The form of the Liapunov function for equation (14) is assumed to be

$$V = \underline{x}_T \underline{S} \underline{x}, \quad (16)$$

where \underline{S} is a symmetric matrix function of \underline{x} . The type of elements in matrix \underline{S} are obtained from the authors' experience with the variable gradient technique. It is for this reason that we place Ku's, Puri's, and Mekel's work in this section of the report. An example of the \underline{S} matrix used by Ku and Puri will be given shortly. In Mekel's thesis [11], a slightly different form of \underline{S} is used in most of his examples.

Similarly, the time derivative of V related to the system in (14) is assumed to take on the form

$$\dot{V} = \underline{x}_T \underline{T} \underline{x}, \quad (17)$$

where \underline{T} is a matrix function to be determined. For the asymptotic stability of $\underline{x} = \underline{0}$, we require that \underline{S} be positive definite and \underline{T} be negative semi-definite,

along with certain closedness properties of V being fulfilled.

The authors consider two formulations for \dot{V} . The first formulation is

$$\begin{aligned}\dot{V} &= \dot{x}_T \underline{X} x + x_T \dot{\underline{S}} x + x_T \underline{S} \dot{x}, \\ &= x_T (\underline{A}_T \underline{S} + \underline{S} \underline{A} + \dot{\underline{S}}) x = x_T \underline{T}_1 x,\end{aligned}\quad (18)$$

where

$$\underline{T}_1 = \underline{A}_T \underline{S} + \underline{S} \underline{A} + \dot{\underline{S}}. \quad (19)$$

The second formulation of \dot{V} is derived by first computing $\nabla V = \underline{B} x$ and then forming

$$\dot{V} = (\nabla V)_T \dot{x} = x_T (\underline{B}_T \underline{A}) x = x_T \underline{T}_2 x, \quad (20)$$

where

$$\underline{T}_2 = \underline{B}_T \underline{A}. \quad (21)$$

The matrices \underline{T}_1 and \underline{T}_2 must be negative semi-definite. To insure that this is the case, the elements of \underline{T} , T_{ij} , must satisfy the following conditions:

$$\begin{aligned}T_{ii} &\leq 0, \text{ for } i = 1, 2, \dots, n, \\ T_{ij} + T_{ji} &= 0, \text{ for } i \neq j \text{ and } i, j = 1, 2, \dots, n.\end{aligned}\quad (22)$$

As we will show later, Mekel in reference [1] verifies that the above formulations give identical V and \dot{V} .

The form of the \underline{S} matrix used by Ku and Puri, in [9], for a fourth order, autonomous system is

$$\underline{S} = \begin{array}{|c|c|c|c|} \hline k_{11} + \frac{y_{11}}{2x_1} & k_{12} + \frac{f_{12}}{2x_1} & k_{13} + \frac{f_{13}}{2x_1} & k_{14} + \frac{f_{14}}{2x_1} \\ \hline k_{12} + \frac{f_{12}}{2x_1} & k_{22} + \frac{y_{22}}{x_2^2} & k_{23} & k_{24} \\ \hline k_{13} + \frac{f_{13}}{2x_1} & k_{23} & k_{33} + \frac{y_{33}}{2x_3} & k_{34} \\ \hline k_{13} + \frac{f_{14}}{2x_1} & k_{24} & k_{34} & \frac{y_{44}}{2x_4} + k_{44} \\ \hline \end{array} \quad (23)$$

where Y_{ii} is an even function defined by

$$Y_{ii} = \int_0^{x_i} y_i(x_i) dx_i, i = 1, 2, 3, 4. \quad (24)$$

Since Y_{ii} is an even function of x_i , then y_i is an odd function of x_i . The k_{ij} 's are constants to be determined and the f_{1i} 's are unknown functions of x_1 . The y_i , k_{ij} and f_{1i} are chosen such that V is a Liapunov function. The B matrix in equation (21) which is derived from (23) has a form nearly like those considered by Schultz and Gibson.

In the following discussion, the equivalence of the two formulations for \dot{V} is presented by considering examples of Mekel.

MEKEL'S WORK [11]

The two formulations for obtaining the time derivative of V , as described above, are now illustrated by a third order nonlinear system with three nonlinearities. In the first the T_1 matrix is derived and in the second formulation the T_2 matrix is derived.

First Formulation

The nonlinear equation is

$$\ddot{x} + \left[a_3 + \phi(x, \dot{x}) \right] \ddot{x} + g(\dot{x}) + f(x) = 0. \quad (25)$$

The matrix A in equation (15) becomes

$$A = \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline -f/x_1 & -g/x_2 & -(a_3 + \phi) \\ \hline \end{array} \quad (26)$$

where $f = f(x_1)$, $g = g(x_2)$ and $\phi = \phi(x_1, x_2)$. The S matrix in equation (16) is given by

$\underline{S} =$

$k_{11} + \frac{Y_1}{x_1^2} + Y$	$k_{12} + f_1/x_1$	k_{13}	(27)
$k_{12} + f_1/x_1$	$k_{22} + \frac{Y_2}{x_2^2} + Y$	k_{23}	
k_{13}	k_{23}	1	

where the k_{ij} 's are constants and $Y_i = Y_i(x_i)$, $f_1 = f_1(x_1)$ and $Y = Y(x_1, x_2)$.

These constants and functions are to be determined such that V is a Liapunov function.

Since \dot{V} is a "quadratic form", the matrix \underline{T}_1 in equation (19) can be replaced by an equivalent \underline{T}_1 defined by the expression $(2 \underline{S} \underline{A} + \dot{\underline{S}})$. Thus, from (26) and (27) we have:

$2 \underline{S} \underline{A} = 2$

$-k_{13} f/x_1$	$k_{11} + \frac{Y_1}{x_1^2} + Y - \frac{k_{13} g}{x_2}$	$k_{12} + \frac{f_1}{x_1} - k_{13}(a_3 + \phi)$
$-k_{23} f/x_1$	$k_{12} + \frac{f_1}{x_1} - k_{23} \frac{g}{x_2}$	$k_{22} + \frac{Y_2}{x_2^2} + Y - k_{23}(a_3 + \phi)$
$-f/x_1$	$k_{13} - g/x_2$	$k_{23} - (a_3 + \phi)$

and

$\dot{\underline{S}} =$

$\frac{Y_1' x_2 + \dot{Y}}{x_1^2} - \frac{2(Y_1 + Y) x_2}{x_1^3}$	$\left[f_1'/x_1 - f_1/x_1^2 \right] x_2$	0
$\left[f_1'/x_1 - f_1/x_1^2 \right] x_2$	$\frac{Y_2' x_3 + \dot{Y}}{x_2^2} - \frac{2(Y_2 + Y) x_3}{x_2^3}$	0
0	0	0

where $"/'$ " is defined as $\frac{d}{dx_1}$ or $\frac{d}{dx_2}$. Again, because \dot{V} is written in quadratic form,

we have an equivalent form for $\dot{\underline{S}}$:

$$\underline{\dot{S}}_e = 2 \quad \begin{array}{|c|c|c|} \hline 0 & 1/x_1 \frac{\partial Y}{\partial x_1} - \frac{Y_1 + Y}{x_1^2} & 0 \\ \hline Y_1' / 2x_1 & f_1' - f_1 / x_1 & 1/x_2 \frac{\partial Y}{\partial x_2} - \frac{Y_2 + Y}{x_2^2} \\ \hline 0 & Y_2' / 2x_2 & 0 \\ \hline \end{array} \quad (30)$$

Combining equations (28) and (30) we form an equivalent \underline{T}_1^* :

$$\underline{T}_1^* = 2 \quad \begin{array}{|c|c|c|} \hline -k_{13} f/x_1 & k_{11} + \frac{1}{x_1} \frac{\partial Y}{\partial x_1} - \frac{k_{13} g}{x_2} & k_{12} + \frac{f_1}{x_1} - k_{13}(a_3 + \phi) \\ \hline \frac{Y_1'}{2x_1} - k_{23} f/x_1 & k_{12} + f_1' - k_{23} g/x_2 & k_{22} + \frac{1}{x_2} \frac{\partial Y}{\partial x_2} - k_{33}(a_3 + \phi) \\ \hline -f/x_1 & k_{13} - g/x_2 + \frac{Y_2'}{2x_2} & k_{23} - (a_3 + \phi) \\ \hline \end{array} \quad (31)$$

Applying conditions (22) to equation (31), the unknown terms of \underline{T}_1^* can be determined. Thus, we let

$$k_{13} = 0. \quad (32)$$

If ϕ is constrained to be a positive function, then element $T_{33} < 0$ when

$$k_{23} = a_3. \quad (33)$$

$$\text{For } T_{13} + T_{31} = 0,$$

$$k_{12} = 0 \quad (34)$$

and

$$f_1 = f \equiv f(x_1). \quad (35)$$

$$\text{For } T_{23} + T_{32} = 0,$$

$$k_{22} = a_3^2, \quad (36)$$

$$\frac{\partial Y_2}{\partial x_2} = 2 g(x_2), \quad (37)$$

and

$$\frac{\partial Y}{\partial x_2} \equiv \left[a_3 x_2 \phi(x_1, x_2) \right]. \quad (38)$$

From (37) and (38), we have by integration

$$Y_2 = 2 \int_0^{x_2} g(x_2) dx_2 = 2G(x_2), \quad (39)$$

and

$$Y = a_3 \int_0^{x_2} x_2 \phi(x_1, x_2) dx_2. \quad (40)$$

$$\text{For } T_{12} + T_{21} = 0,$$

$$k_{11} = 0, \quad (41)$$

and

$$\frac{\partial Y_1}{\partial x_1} = 2a_3 f(x_1). \quad (42)$$

Integrating equation (42) gives

$$Y_1 = 2a_3 \int_0^{x_1} f(x_1) dx_1 = 2a_3 F(x_1). \quad (43)$$

The remaining term, $\frac{1}{x_1} \frac{\partial Y}{\partial x_1}$, in T_{12} can be obtained from (40):

$$\frac{1}{x_1} \frac{\partial Y}{\partial x_1} = \frac{a_3}{x_1} \int_0^{x_2} x_2 \frac{\partial \phi(x_1, x_2)}{\partial x_1} dx_2. \quad (44)$$

Therefore, an equivalent matrix for T_1^* can be written as:

$$\underline{T} = 2 \begin{array}{|c|c|c|} \hline 0 & \frac{a_3}{x_1} \int_0^{x_2} \frac{\partial \phi(x_1, x_2)}{\partial x_1} x_2 dx_2 & 0 \\ \hline 0 & - \left[a_3 \frac{g(x_2)}{x_2} - f'(x_1) \right] & 0 \\ \hline 0 & 0 & - \phi(x_1, x_2) \\ \hline \end{array} \quad (45)$$

where $\dot{V} = \underline{x}_T \underline{T} \underline{x}$. Matrix \underline{S} in equation (27) becomes

$$\underline{S} = \begin{array}{|c|c|c|} \hline \frac{Y_1 + Y}{x_1^2} & f(x_1)/x_1 & 0 \\ \hline f(x_1)/x_1 & a_3^2 + \frac{Y_2 + Y}{x_2^2} & a_3 \\ \hline 0 & a_3 & 1 \\ \hline \end{array}, \quad (46)$$

where Y_2 , Y and Y_1 , are defined by (39), (40), and (43) respectively. Since $V = \underline{x}_T \underline{S} \underline{x}$, we have

$$V = 2a_3 F(x_1) + 2 f(x_1) x_2 + 2G(x_2) + 2a_3 \int_0^{x_2} \phi(x_1, x_2) x_2 dx_2 + (a_3 x_2 + x_3)^2. \quad (47)$$

From (45), we have the time derivative of V ,

$$\dot{V} = -2 \left[a_3 g(x_2)/x_2 - f'(x_1) \right] x_2^2 - 2 \phi(x_1, x_2) x_2^2 + a_3 x_2 \int_0^{x_2} \left[\frac{\partial \phi(x_1, x_2)}{\partial x_1} \right] x_2 dx_2. \quad (48)$$

For $V > 0$, we require that

$$F(x_1) > 0, \text{ or } x_1 f(x_1) > 0, \quad (49)$$

$$G(x_2) > 0, \text{ or } x_2 g(x_2) > 0, \quad (50)$$

$$2a_3 F(x_1) + 2f(x_1) x_2 + 2G(x_2) > 0, \quad (51)$$

and

$$\phi(x_1, x_2) \geq 0. \quad (52)$$

For $\dot{V} \leq 0$, we require that

$$\left[a_3 g(x_2)/x_2 - f'(x_1) \right] > 0, \quad (53)$$

$$\phi(x_1, x_2) x_2^2 \geq \sigma^2 > 0, \quad (54)$$

and

$$a_3 \int_0^{x_2} \frac{\partial \phi(x_1, x_2)}{\partial x_1} x_2 dx_2 < \delta. \quad (55)$$

As long as the closedness properties are satisfied, equations (49) to (55) give the conditions required for asymptotic stability.

Second Formulation

Using the \underline{S} matrix in (27), we can write $V = \underline{x}_T \underline{S} \underline{x}$ as

$$\begin{aligned} V = & k_{11} x_1^2 + Y_1 + 2 k_{12} x_1 x_2 + 2 f_1 x_2 + \\ & + 2 k_{13} x_1 x_3 + k_{22} x_2^2 + Y_2 + 2 k_{23} x_2 x_3 + \\ & + x_3^2 + 2 Y. \end{aligned} \quad (56)$$

Partial differentiation of V gives the following components for ∇V :

$$\begin{aligned} \frac{\partial V}{\partial x_1} = & 2 k_{11} x_1 + \frac{dY_1}{dx_1} + 2 \frac{\partial Y}{\partial x_1} + 2 k_{12} x_2 + 2 \frac{df_1}{dx_1} x_2 + \\ & + 2 k_{13} x_3, \end{aligned} \quad (57)$$

$$\begin{aligned} \frac{\partial V}{\partial x_2} = & 2 k_{12} x_1 + 2 f_1 + 2 k_{22} x_2 + \frac{dY_2}{dx_2} + 2 \frac{dY}{dx_2} + \\ & + 2 k_{23} x_3, \end{aligned} \quad (58)$$

$$\frac{\partial V}{\partial x_3} = 2 k_{13} x_1 + 2 k_{23} x_2 + 2 x_3. \quad (59)$$

If we now express the gradient of V in matrix form, we have

$$\nabla V = \underline{B} \underline{x} = 2 \begin{array}{|c|c|c|} \hline k_{11} + \frac{Y_1'}{2x_1} + \frac{1}{x_1} \frac{\partial Y}{\partial x_1} & k_{12} + f_1' & k_{13} \\ \hline k_{12} + f_1/x_1 & k_{22} + \frac{Y_2'}{2x_2} + \frac{1}{x_2} \frac{\partial Y}{\partial x_1} & k_{23} \\ \hline k_{13} & k_{23} & 1 \\ \hline \end{array} \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline x_3 \\ \hline \end{array}. \quad (60)$$

Combining equations (21), (26), and (60) gives

$$\underline{T} = 2 \begin{array}{|c|c|c|} \hline -k_{13} f/x_1 & k_{11} + \frac{Y_1}{2x_1} + \frac{1}{x_1} \frac{\partial Y}{\partial x_1} - k_{13} g/x_2 & k_{12} + f_{1/x_1} - k_{13}(a_3 + \phi) \\ \hline -k_{23} f/x_1 & k_{12} + f_1' - k_{23} g/x_2 & k_{22} + \frac{Y_2}{2x_2} + \frac{1}{x_2} \frac{\partial Y}{\partial x_2} - k_{23}(a_3 + \phi) \\ \hline -f/x_1 & k_{13} - g/x_2 & k_{23} - (a_3 + \phi) \\ \hline \end{array} \quad (61)$$

Applying conditions (22) to equation (61) allows us to determine the unknown elements of the \underline{T}_2 matrix. As in the first formulation, if we start with

$$k_{13} = 0, \phi > 0, k_{23} = a_3, \quad (62)$$

and then proceed in a similar fashion as before, we arrive at the same simplified \underline{T} matrix as given in (45). Hence, V and \dot{V} are the same as obtained by the first formulation and the stability conditions are also the same. Actually, if we consider (31) and (61) very little difference in the matrices is seen and in fact, the two matrices give the same quadratic form $\dot{V} = \underline{x}_T \underline{T} \underline{x}$.

Mekel also shows in reference [11] that the first and second formulations lead to the same stability conditions for a fourth order system with three nonlinearities. This work will not be repeated here, but it will occur in the compendium of examples at the end of this section.

PURI'S WORK [12]

Since this material is practically the same as given in the previous discussions, we will only briefly outline the procedure. Consider the nonautonomous, nonlinear system given by

$$\dot{\underline{x}} = \underline{A}(\underline{x}, t) \underline{x}, \quad (63)$$

where \underline{A} is the same form as given in (15) but now $a_i = a_i(\underline{x}, t)$. The equilibrium solution of (63) is taken as $\underline{x} = \underline{Q}$. The choice for a Liapunov function is

$$V = \underline{x}_T \underline{S} (\underline{x}, t) \underline{x}, \quad (64)$$

where \underline{S} is symmetric and the elements are

$$S_{ii} = \frac{f_{ii}(x_i)}{2 x_i}, \quad i = 1, 2, \dots, n-1,$$

$$S_{nn} = 1, \quad (65)$$

$$S_{ij} = \frac{f_{ij}(x_i, t)}{x_i}, \quad i \neq j; i, j = 1, 2, \dots, n.$$

The time derivative of V is

$$\dot{V} = (\nabla V)_T \underline{A} \underline{x} + \underline{x}_T \dot{\underline{S}} \underline{x} = \underline{x}_T (\underline{B}_T \underline{A} + \dot{\underline{S}}) \underline{x} = \underline{x}_T \underline{T} \underline{x}, \quad (66)$$

where $\underline{T} = \underline{B}_T \underline{A} + \dot{\underline{S}}$ and $(\nabla V)_T = \underline{x}_T \underline{B}_T$. The \underline{B} matrix in $(\nabla V)_T$ is of the form:

$$\underline{B} = \begin{array}{ccccc} f'_{11/x_1} & 2 f'_{12} & . & . & . & 2 f_{1,n-1} & 2 f'_{1n} \\ 2 f'_{12/x_1} & f'_{22/x_2} & . & . & . & 2 f'_{2,n-1} & 2 f'_{2n} \\ 2 f_{13/x_1} & 2 f_{23/x_2} & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & \frac{f'_{n-1,n-1}}{x_{n-1}} & 2 f'_{n-1,n} \\ 2 f_{1n/x_1} & 2 f_{2n/x_2} & . & . & . & 2 \frac{f_{n-1,n}}{x_{n-1}} & 2 \end{array} \quad (67)$$

The arbitrary functions, f_{ij} , are chosen such that elements, T_{ij} , of \underline{T} satisfy the conditions

$$T_{ii} \leq 0 \quad \text{and} \quad T_{ij} + T_{ji} = 0. \quad (68)$$

Thus, \underline{T} is negative semi-definite and the elements of \underline{S} are known. The stability conditions are then obtained from the requirement that \underline{S} be positive definite and from those conditions imposed by equation (68).

WORK OF HO, GOLDWYN AND NARENDRA, [13], [14], [15]

The authors considered the following systems:

$$\begin{aligned}\dot{\underline{x}} &= \underline{f}(\underline{x}), \\ \underline{x}(0) &= \underline{c}, \\ \underline{f}(0) &= \underline{o},\end{aligned}\tag{69}$$

where \underline{c} is any initial vector. The only equilibrium solution that the above systems have is the null solution, $\underline{x} = \underline{o}$. Also, the authors only considered asymptotic stability in the large, (ASL); that is, $\underline{x}(t) \rightarrow \underline{o}$ as $t \rightarrow \infty$ for any \underline{c} . The Liapunov functions used to analyze the nonlinear systems are generated from common Liapunov functions, (CLF), which are derived for various types of linear systems.

The authors generated the CLF's in various ways. In [14], a CLF for a linear, non-feedback system and a linear, feedback system was obtained. The results were then used to analyze a nonlinear, feedback system. Also, in [14], the CLF concept was generalized in order to be applicable to a nonlinear system of the form $\dot{\underline{x}} = \underline{A}(\underline{x}) \underline{x}$. In [15], a CLF was defined for a linear, variable parameter system of the form $\underline{x} = \underline{A}(K_1, K_2, \dots, K_m) \underline{x}$, where K_1, K_2, \dots, K_m are system parameters. Using these results, Liapunov functions for certain nonlinear equations were generated.

All of these problems assume that $\dot{\underline{V}} = -\underline{x}_T \underline{Q} \underline{x}$ is known and we search for a matrix \underline{P} such that $V = \underline{x}_T \underline{P} \underline{x}$ is positive definite. This is the approach of the variable gradient method. The difference here being the technique used to obtain \underline{P} given the matrix \underline{Q} . We now present the various problems considered by the authors.

Let's consider the feedback problem. The linear system with no feedback is defined by

$$\dot{\underline{x}} = \underline{A} \underline{x}, \quad \underline{A} \equiv \text{constant};\tag{70}$$

and the linear system with feedback is defined by

$$\dot{\underline{x}} = \underline{A} \underline{x} + \underline{d} m \quad (71)$$

where \underline{c} and \underline{d} are constant vectors and

$$m = \underline{c}_T \underline{x} . \quad (72)$$

We assume that both systems (70) and (71) are ASL. If a CLF for both (70) and (71) exists, then the form of V is given by

$$V = \underline{x}_T \underline{P} \underline{x} , \quad (73)$$

where \underline{P} is a constant matrix. The time derivative of V corresponding to (70) is

$$\dot{V} = \underline{x}_T \left[\underline{A}_T \underline{P} + \underline{P} \underline{A} \right] \underline{x} = - \underline{x}_T \underline{Q}_1 \underline{x} , \quad (74)$$

where \underline{Q}_1 is positive definite. The time derivative of V corresponding to (71) is

$$\begin{aligned} \dot{V} &= \underline{x}_T \left[\underline{A} + \underline{d} \underline{c}_T \right] \underline{P} \underline{x} + \underline{x}_T \underline{P} \left[\underline{A} \underline{x} + \underline{d} \underline{c}_T \right] \underline{x} \\ &= - \underline{x}_T \underline{Q}_2 \underline{x} , \end{aligned} \quad (75)$$

where

$$- \underline{Q}_2 = - \underline{Q}_1 + \underline{P} \underline{d} \underline{c}_T + \underline{c} \underline{d}_T \underline{P} . \quad (76)$$

Thus, \underline{Q}_2 is positive definite if

$$m \underline{x}_T \underline{P} \underline{d} \leq 0 , \quad (77)$$

where $m = \underline{c}_T \underline{x}$. Now we introduce the following nonlinearity into the feedback system:

$$0 \leq \frac{f(m)}{m} = \frac{u}{m} < 1 , \quad (78)$$

where $u = f(m)$, the system equation becomes

$$\dot{\underline{x}} = \underline{A} \underline{x} + \underline{d} u . \quad (79)$$

As a candidate for the Liapunov function of (79), we consider the form given in (73). The time derivative of V corresponding to (79) is given as

$$\dot{V} = \underline{x}_T (\underline{A}_T \underline{P} + \underline{P} \underline{A}) \underline{x} + 2u \underline{x}_T \underline{P} \underline{d} . \quad (80)$$

This nonlinear system is ASL since from (77), (78) and (80) we have:

$$u \underline{x}_T \underline{P} \underline{d} < m \underline{x}_T \underline{P} \underline{d} \leq 0. \quad (81)$$

Thus, if there exists a CLF for systems (70) and (71), then for a nonlinear feedback system defined by (78) and (79) there exists a Liapunov function which guarantees the ASL of the system. Namely, the same CLF as given for the linear systems. With suitable restrictions on the vector function, $\underline{f}(\underline{x})$, this analysis can be used for stability studies of the system defined by

$$\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{d} u. \quad (82)$$

Also, in reference [14], the CLF concept was generalized such that the nonlinear system

$$\dot{\underline{x}} = \underline{A}(\underline{x}) \underline{x} \quad (83)$$

could be analyzed. We begin this analysis by considering a linear system.

That is, for the linear system

$$\dot{\underline{x}} = \underline{A} \underline{x}, \underline{A} \equiv \text{constant}, \quad (84)$$

the necessary and sufficient condition for ASL is that for any positive definite matrix \underline{Q} there exists a positive definite matrix \underline{P} such that

$$\underline{Q} = -(\underline{A}_T \underline{P} + \underline{P} \underline{A}). \quad (85)$$

The solution of (85), given \underline{Q} and \underline{A} , is

$$\underline{P} = - \int_0^\infty e^{(\underline{A}_T)t} \underline{Q} e^{(\underline{A})t} dt, \quad (86)$$

if $\text{Re}(\lambda_K(\underline{A})) < 0$ for all the distinct eigenvalues of \underline{A} , $\lambda_K(\underline{A})$. We now let $\{\underline{Q}\}$ be the set of all positive definite matrices \underline{Q} . The corresponding set $\{\underline{P}\}$ is a subset of $\{\underline{Q}\}$ and is the image of the mapping of set $\{\underline{Q}\}$ into itself under the transformation defined by

$$\{\underline{Q}\} = -(\underline{A}_T \underline{P} + \underline{P} \underline{A}), \quad (87)$$

or

$$\{\underline{P}\} = - \int_0^{\infty} e^{\underline{A}^T t} \{Q\} e^{\underline{A} t} dt, \quad (88)$$

where \underline{A} is defined by (84). If $\operatorname{Re}(\lambda_K(\underline{A})) < 0$ for all K , then the mapping is unique.

Using the above results from linear systems, the sufficient conditions for ASL of the nonlinear system (83) can be derived. We let \underline{x}_0 be any fixed vector in the whole \underline{x} - space and define $\underline{A}(\underline{x}_0)$ to be \underline{A}_0 , the set $\{\underline{P}_0\}$ are the \underline{P} 's corresponding to \underline{A}_0 and defined by (88). The resulting conditions for ASL are:

- (i) $\operatorname{Re}(\lambda_K(\underline{A}_0)) < 0$ for all \underline{x}_0 and K .
- (ii) The intersection of all the sets $\{\underline{P}_0\}$ is not empty.

Thus, from this nonempty intersection there exists a matrix \underline{P} which gives a Liapunov function for all \underline{x}_0 ; namely

$$V = \underline{x}_T \underline{P} \underline{x}, \text{ and}$$

$$\dot{V} = \underline{x}_T (\underline{A}_T(\underline{x}_0) \underline{P} + \underline{P} \underline{A}(\underline{x}_0)) \underline{x} < 0.$$

Therefore, $\dot{\underline{x}} = \underline{A}(\underline{x}) \underline{x}$ is ASL and the Liapunov function used in the analysis is a CLF of the system $\dot{\underline{x}} = \underline{A}(\underline{x}_0) \underline{x}$.

In reference [13], the authors considered the linear, varying parameter system

$$\dot{\underline{x}} = \underline{A}(K) \underline{x}, \quad (89)$$

where $\underline{K} \leq K \leq \bar{K}$. $\underline{A}(K)$ is assumed to be linearly dependent upon the parameter K .

We choose the same form for V as in the previous cases $\underline{x}_T \underline{P} \underline{x}$, where

$\dot{V}(\underline{x}) = - \underline{x}_T \underline{Q} \underline{x}$. The \underline{Q} matrix satisfies equation (85). Since for any symmetric,

positive definite matrix \underline{Q} there exists a unique solution of (85), then the above

V -function guarantees the asymptotic stability of (89) in a given range for the

parameter K . Since \underline{A} depends on K , \underline{P} is also dependent upon K through equation (85).

Defining set $\{\underline{P}\}$ as above, the CLF for (89) is defined as: "if there exists a \underline{P}^* belonging to set $\{\underline{P}\}$ such that $Q^* = - \begin{bmatrix} \underline{A}_T & \underline{P}^* \\ \underline{P}^* & \underline{A} \end{bmatrix}$ is positive definite for all K in the range $\underline{K} \leq K \leq \bar{K}$, then $V^* = \underline{X}_T \underline{P}^* \underline{X}$ is a CLF over $\underline{K} \leq K \leq \bar{K}$."

By using this CLF, nonlinear systems defined by replacing K by $K(\underline{x})$ can be studied. To aid in the discussion, we consider the first and second order examples given in [15].

First Order System, [13]

The system is described by the equation

$$\dot{x} = ax - Kx, \quad (90)$$

where K is a system parameter. The Liapunov function of the form given by (73) is

$$V = px^2. \quad (91)$$

Then

$$\dot{V} = 2px \left[ax - Kx \right] = 2p (a - K) x^2 = -qx^2. \quad (92)$$

Equation (90) is ASL for $K > a$ and for $p > 0$. A CLF exists if the parameter K is restricted by the inequalities

$$K \geq \underline{K} > a. \quad (93)$$

If we choose

$$p = \underline{K} - a, \quad (94)$$

then $V = (\underline{K} - a) x^2$ is a CLF for the system in (90) where the parameter K satisfies (93).

If K is now replaced by a nonlinear function $K(x)$, we can still use $V = (\underline{K} - a)x^2$, as a CLF, provided $\underline{K} = \min_x \left[K(x) \right] > a$. The time derivative of this V is

$$\dot{V} = 2 (\underline{K} - a) (a - K(x)) x^2 < 0. \quad (95)$$

Therefore the nonlinear equation

$$\dot{x} = ax - K(x) x, \quad (96)$$

is ASL for $K > a$. This example shows how a CLF for a linear system can be used for a nonlinear system.

Second-Order Example 15

Consider the second order example

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -K(x_1) x_1 - ax_2\end{aligned}\tag{97}$$

where $K(x_1) = b + f(x_1)$. In terms of equation (83),

$$\underline{A}(\underline{x}) = \begin{bmatrix} 0 & 1 \\ -K(x_1) & -a \end{bmatrix},\tag{98}$$

where K is initially treated as a parameter and the range of interest is $K \geq 0$. If the candidate for V is given by $V = \underline{x}_T \underline{P} \underline{x}$, then $\dot{V} = -\underline{x}_T \underline{Q} \underline{x}$, where

$$\underline{Q}(K) = \begin{bmatrix} 2 K p_{12} & K p_{22} - p_{11} + a p_{12} \\ K p_{22} - p_{11} + a p_{12} & 2 a p_{22} - 2 p_{12} \end{bmatrix},\tag{99}$$

and P_{ij} are the elements of \underline{P} . If there exists a CLF, then there corresponds a \underline{P} for $\underline{Q}(0)$; namely, for

$$\underline{Q}(0) = \begin{bmatrix} 0 & a p_{12} - p_{11} \\ a p_{12} - p_{11} & 2 (a p_{22} - p_{12}) \end{bmatrix}.\tag{100}$$

For $\underline{Q}(0)$ to be positive semi-definite, we must select $a p_{12} = p_{11}$. Hence, we have

$$\underline{Q}(K) = \begin{bmatrix} 2 K p_{12} & K p_{22} \\ K p_{22} & 2 (a p_{22} - p_{12}) \end{bmatrix}.\tag{101}$$

For positive definiteness of $Q(K)$ it is necessary that

$$\begin{aligned} (i) \quad & p_{12} > 0, \quad ap_{22} - p_{12} > 0, \\ (ii) \quad & \det Q(K) = q(K) > 0, \\ & q(K) = -K^2 p_{22}^2 + 4Kp_{12} (ap_{22} - p_{12}). \end{aligned} \quad (102)$$

Since $q(K)$ can be negative, as seen by (102), there exists a maximum value of K .

Thus, the maximum is

$$\bar{K} = \left[\frac{4 p_{12} (ap_{22} - p_{12})}{p_{22}^2} \right]; \quad (103)$$

and a CLF exists for all K in the interval $0 \leq K \leq \bar{K}$. (Any matrix P which satisfies $ap_{12} = p_{11}$ and (102) is positive definite if $K > 0$ and $a > 0$.) If we can choose a CLF for the range $0 \leq K(x) \leq \bar{K}$, then the nonlinear system in (97) can be proved to be ASL. This choice for V is

$$V = \underline{x}_T \begin{array}{|c|c|} \hline a^2 & a \\ \hline a & 2 \\ \hline \end{array} \underline{x}, \quad (104)$$

where $\bar{K} = a^2$, by (103).

The disadvantage of the above procedure is that for the linear problem corresponding to (97), $K(x_1) = \text{constant}$, there is an upper bound on K , given by (103). This can be corrected if P_{11} in \underline{P} is written as $p_{11} = c + vK$, where c and v are constants. Thus from $Q(0)$ in (99) and the conditions (102), we have that $c = ap_{12}$, $v = p_{22}$ and $q(K) > 0$ for all $K > 0$. Now, if we further choose $\begin{bmatrix} p_{22} = 2 \end{bmatrix}$ and $\begin{bmatrix} p_{12} = a \end{bmatrix}$, then \underline{P} becomes

$$\underline{P} = \begin{array}{|c|c|} \hline a^2 + 2K & a \\ \hline a & 2 \\ \hline \end{array} \quad (105)$$

Therefore, $Q=Q(K)$ and $\underline{P}=\underline{P}(K)$ are both positive definite for all $K > 0$. Thus,

for the linear system, $K=\text{constant}$ in (97), the Liapunov function is

$$V(\underline{x}) = \underline{x}_T \begin{array}{|c|c|} \hline a^2 & a \\ \hline a & 2 \\ \hline \end{array} \underline{x} + 2 K x_1^2 \quad (106)$$

$$= \underline{x}_T \underline{P}_0 \underline{x} + 2 K x_1^2 ,$$

where \underline{P}_0 is a constant matrix independent of K . Equation (106) is a modification of the CLF in (104) because of the added term $2Kx_1^2$.

For nonlinear systems, an idea due to Cartwright is applied, in that the term $2Kx_1^2$ is replaced by

$$4 \int_0^{x_1} u K(u) du . \quad (107)$$

Thus, for the nonlinear system in (97) the modified CLF is:

$$V(\underline{x}) = \underline{x}_T \underline{P}_0 \underline{x} + 4 \int_0^{x_1} u K(u) du \quad (108)$$

and

$$\dot{V}(\underline{x}) = - \underline{x}_T \underline{Q}(\underline{x}) \underline{x}, \quad (109)$$

where

$$\underline{Q} = \begin{array}{|c|c|} \hline 2aK(x_1) & 0 \\ \hline 0 & 2a \\ \hline \end{array} .$$

This Liapunov function in (108) proves that (97) is ASL if $K(x_1) > 0$.

Generalization

The above results are generalized to a system with several parameters K_1, K_2, \dots, K_m . For the linear system, the form of the candidate for the Liapunov function is

$$V(\underline{x}) = \underline{x}_T \underline{P}_0 \underline{x} + \sum_{i=1}^m V_i K_i x_i^2 \quad (110)$$

where \underline{P}_0 is independent of the K_i 's, and V_i 's are constants ($i = 1, 2, \dots, m, m \leq n$). For nonlinear systems, if $K_i = K_i(x_i)$, the Liapunov function may be modified to

$$V(\underline{x}) = \underline{x}^T \underline{P}_0 \underline{x} + 2 \sum_{i=1}^m V_i \int_0^{x_i} K_i(u) u du. \quad (111)$$

The time derivative of V gives identical forms for Q in both (110) and (111).

Since $V(\underline{x})$ in (111) can be bounded in the following way

$$V(\underline{x}) \geq \underline{x}^T \underline{P}_0 \underline{x} + \sum_{i=1}^m V_i K_i^* x_i^2, \quad (112)$$

where

$$K_i^* = \begin{cases} \max_{x_i} [K_i(x_i)] & , V_i < 0 \\ \min_{x_i} [K_i(x_i)] & , V_i > 0 \end{cases} \quad (113)$$

then $V(\underline{x})$ in (111) is positive ($\|\underline{x}\| \neq 0$) if $V(\underline{x})$ in (110) is positive for all K_i such that

$$\min_{x_i} [K_i(x_i)] \leq K_i \leq \max_{x_i} [K_i(x_i)]. \quad (114)$$

Summary:

The results of the last few paragraphs may be summarized as follows:

Consider the differential equation $\dot{\underline{x}} = \underline{A} \underline{x}$ which depends linearly on the parameters K_1, K_2, \dots, K_m . If the solution of the system is stable for $\underline{K}_i \leq K_i \leq \bar{K}_i$, then $V = \underline{x}^T \underline{P} (K_1, \dots, K_m) \underline{x}$ is a Liapunov function if \underline{P} is the solution of the equation

$$\underline{A}^T (K_1, \dots, K_m) \underline{P} + \underline{P} \underline{A} (K_1, \dots, K_m) = -\underline{Q} (K_1, \dots, K_m). \quad (115)$$

If a " $\underline{Q} > 0$ " can be selected such that \underline{P} is independent of the K_i 's, $V(\underline{x}) = \underline{x}^T \underline{P} \underline{x}$

is a CLF in the range $\underline{K}_i \leq K_i \leq \bar{K}_i$ and, consequently, for the nonlinear system where $K_i = K_i(\underline{x})$. (The trouble is that such CLF's are not easy to get.) If the nonlinear system is such that $K_i = K_i(x_i)$, then a Liapunov function of the form given in (111) can be obtained. This Liapunov function is greater than zero ($\|\underline{x}\| \neq 0$) if $\text{MAX}_{x_i} [K_i(x_i)] = \bar{K}_i$ and $\text{MIN}_{x_i} [K_i(x_i)] = \underline{K}_i$, as can be seen from (112) and (113). Thus, the method considered by the authors in the above discussion is one in which the nonlinear problem is related to the corresponding linear problem for which the stability conditions are known.

COMPENDIUM OF EXAMPLES

The following set of examples were taken from the paper of Schultz and Gibson, [1] .

(1) Second Order System

$$\dot{x}_1 = x_2 ,$$

$$\dot{x}_2 = -x_2 - x_1^3 .$$

The gradient is assumed to be

$$\nabla V = \begin{array}{|c|c|} \hline \alpha_{11} & \alpha_{12} \\ \hline \alpha_{21} & 2 \\ \hline \end{array} \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \end{array} ,$$

where the α 's are constants or functions of \underline{x} . The time derivative becomes

$$\dot{V} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{array}{|c|c|} \hline \alpha_{11} & \alpha_{21} \\ \hline \alpha_{12} & 2 \\ \hline \end{array} \begin{array}{|c|} \hline x_2 \\ \hline -x_2 - x_1^3 \\ \hline \end{array}$$

$$= -\alpha_{21}x_1^4 + (\alpha_{12} - 2)x_1^2 + (\alpha_{11} - \alpha_{21} - 2x_1^2)x_1x_2 .$$

We now let $(\alpha_{11} - \alpha_{21} - 2x_1^2) = 0$, $\alpha_{12} = 1$ and $\alpha_{21} > 0$. Thus,

$\dot{V} < 0$ if $\|\underline{x}\| \neq 0$. And, the gradient becomes

$$\nabla V = \begin{array}{|c|c|} \hline \alpha_{21} + 2x_1^2 & 1 \\ \hline \alpha_{21} & 2 \\ \hline \end{array} \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \end{array} .$$

The curl equations give the following relationships:

$$\frac{\partial V}{\partial x_1} = \alpha_{21} x_1 + 2x_1^3 + x_2 ,$$

$$\frac{\partial V}{\partial x_2} = \alpha_{21} x_1 + 2x_2 ,$$

$$\frac{\partial^2 V}{\partial x_1 \partial x_2} = 1 ,$$

$$\frac{\partial^2 V}{\partial x_2 \partial x_1} = \alpha_{21} .$$

Thus, we let $\alpha_{21} = 1$. The resulting V and \dot{V} are:

$$\begin{aligned} V &= \int \nabla V \cdot d\mathbf{x} = \int_0^{x_1} (x_1 + 2x_1^3) dx_1 + \int_0^{x_2} (x_1 + 2x_2) dx_2, \\ &= x_1^2/2 + x_1^4/2 + x_1x_2 + x_2^2, \\ &= 1/2 [x_1 + x_2]^2 + 1/2 [x_1^4 + x_2^2], \\ \dot{V} &= (\nabla V)_T \dot{\mathbf{x}} = -x_2^2 - x_1^4. \end{aligned}$$

We can see that V is positive definite and \dot{V} is negative definite. Also, as

$\|\mathbf{x}\| \rightarrow \infty$, $V \rightarrow \infty$. Therefore the system is globally asymptotically stable.

(2) Second Order System

This system is described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - f_1x_2 - f_1' x_1x_2 - \beta x_1f_1, \end{aligned}$$

where $f_1 = f(x_1)$, $f_1' = df(x_1)/dx_1$ and $\beta \equiv \text{constant}$. The gradient is chosen to be

$$\nabla V = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The time derivative of V becomes

$$\begin{aligned} \dot{V} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_2 - f_1x_2 - f_1' x_1x_2 - \beta x_1f_1 \end{bmatrix}, \\ &= x_1x_2 \left[\alpha_{11} - \alpha_{21} - \alpha_{21} x_1 f_1' - \alpha_{21} f_1 - 2\beta f_1 \right] + \\ &\quad + x_2^2 \left[\alpha_{12} - 2 \right] - \beta \alpha_{21} f_1 x_1^2 + \\ &\quad - 2f_1 x_2^2 - 2x_1x_2^2 f_1'. \end{aligned}$$

We now let the coefficient of $x_1 x_2$ be zero, $\alpha_{12} = 2$, $\beta > 0$ and $f_1 > 0$ for all x_1 . The gradient becomes

$$\underline{\nabla V} = \begin{array}{|c|c|} \hline \alpha_{21}(1 + f_1 + x_1 f_1') + 2\beta f_1 & 2 \\ \hline \alpha_{21} & 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \end{array}.$$

Applying the curl equations to $\underline{\nabla V}$ gives

$$\frac{\partial^2 V}{\partial x_1 \partial x_2} = 2,$$

$$\frac{\partial^2 V}{\partial x_2 \partial x_1} = \alpha_{21}.$$

Then, if $\alpha_{21} = 2$, the resulting V and \dot{V} functions are

$$\begin{aligned} V &= \int_0^{x_1} 2 \left[x_1 + x_1 f_1 + x_1^2 f_1' + \beta x_1 f_1 \right] dx_1 + \int_0^{x_2} 2 \left[x_1 + x_2 \right] dx_2, \\ &= \left[x_1^2 + 2x_1 x_2 + x_2^2 \right] + 2 \int_0^{x_1} x_1 \left[f_1 + x_1 f_1' \right] dx_1 + \\ &\quad + 2\beta \int_0^{x_1} x_1 f(x_1) dx_1, \\ \dot{V} &= (\underline{\nabla V})_T \dot{\underline{x}} = -2 \left[\left[f_1 + x_1 f_1' \right] x_2^2 + \beta f_1 x_1^2 \right]. \end{aligned}$$

The V -function $\longrightarrow \infty$ as $\| \underline{x} \| \longrightarrow \infty$ and the system is globally asymptotically stable when

$$f_1 + x_1 f_1' \equiv f(x_1) + x_1 f'(x_1) \geq 0,$$

$$f_1 \equiv f(x_1) > 0,$$

$$\beta > 0.$$

(3) A General Second Order Equation

This system is given by

$$\ddot{x} + a(x, \dot{x}) \dot{x} + b(x) x = 0,$$

where a and b are such that a unique solution exist for given initial values and the equilibrium solution is at $\dot{x} = 0$. If $x_1 = x$ and $x_2 = \dot{x}$, then the state-variable equations become

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -a(x_1, x_2) x_2 - b(x_1) x_1.$$

As in the previous cases, the gradient is chosen as

$$\nabla V = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The time derivative of V is

$$\begin{aligned} \dot{V} &= \underline{x}_t \begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & 2 \end{bmatrix} \dot{\underline{x}}, \\ &= \left[\alpha_{11} x_1 + \alpha_{12} x_2, \alpha_{21} x_1 + 2x_2 \right] \begin{bmatrix} x_2 \\ -ax_2 - bx_1 \end{bmatrix} \\ &= \left[\alpha_{11} - a\alpha_{21} - 2b \right] x_1 x_2 + \alpha_{12} x_2^2 - 2ax_2^2 - b\alpha_{21} x_1^2. \end{aligned}$$

Let $\alpha_{11} = a\alpha_{21} + 2b$, $\alpha_{12} = \alpha_{21} = 0$. Thus,

$$\nabla V = \begin{bmatrix} 2b & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and

$$\frac{\partial V}{\partial x_1} = 2b(x_1) x_1,$$

$$\frac{\partial V}{\partial x_2} = 2x_2,$$

$$\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial^2 V}{\partial x_2 \partial x_1} = 0.$$

Therefore the curl equations are satisfied and we have

$$\begin{aligned} V &= 2 \int_0^{x_1} b(x_1) x_1 dx_1 + \int_0^{x_2} 2x_2 dx_2, \\ &= 2 \int_0^{x_1} b(x_1) x_1 dx_1 + x_2^2, \end{aligned}$$

and

$$\dot{V} = -2a(x_1, x_2) x_2^2.$$

The system is globally asymptotically stable if

$$b(x_1) > 0 \text{ for all } x_1,$$

$$\int_0^{x_1} b(x_1) x_1 dx_1 \longrightarrow \infty \text{ as } x_1 \longrightarrow \infty,$$

$$a(x_1, x_2) \geq 0 \text{ for all } x_1 \text{ and } x_2.$$

(4) Third Order System

The equations for the system are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -3x_3 - 2x_2 - 3x_1^2 x_2 - \beta x_1^3.$$

After the usual choice for \underline{V} , the time derivative \dot{V} can be written as

$$\begin{aligned} \dot{V} &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \\ &= (d_{11} - d_{31} - 3d_{31}x_1^2 - \beta d_{32}x_1^2) x_1 x_2 + \\ &\quad + (d_{13} + d_{22} - 3d_{32} - 4 - 6x_1^2) x_2 x_3 + \\ &\quad + (d_{21} - 3d_{31} - 2\beta x_1^2) x_1 x_3 + (d_{23} - 6) x_3^2 + \\ &\quad + (d_{12} - 2d_{32}) x_2^2 - 3d_{32} (x_1 x_2)^2 - \beta d_{31} x_1^4. \end{aligned}$$

There are a large number of ways to constrain \dot{V} . The authors initially set $d_{32} = 0$. To eliminate nonnegative x_2^2 - terms, d_{12} must be zero. Then, all terms in \dot{V} containing x_2 are eliminated. This, together with the curl equations give the following:

$$d_{21} = 0, \quad d_{23} = 0, \quad d_{12} = 0,$$

$$d_{13} = 6x_1^2, \quad d_{22} = 4, \quad d_{31} = 2x_1^2,$$

$$d_{11} = 2d_{31} + 3d_{31} x_1^2 = 4x_1^2 + 6x_1^4.$$

The resulting ∇V becomes

$$\nabla V = \begin{array}{|c|} \hline 6x_1^5 + 4x_1^3 + (6x_1^2) x_3 \\ \hline 4x_2 \\ \hline 2x_1^3 + 2x_3 \\ \hline \end{array}.$$

The corresponding V and \dot{V} are

$$V = \int_0^{\underline{x}} \nabla V \cdot d\underline{x} = x_1^6 + x_1^4 + 2x_2^2 + (2x_1^3) x_3 + x_3^2,$$

$$= (x_1^3 + x_3)^2 + x_1^4 + 2x_2^2,$$

$$\dot{V} = -2\beta x_1^6 - (2\beta + 6) x_1^3 x_3 - 6x_3^2.$$

Thus, \dot{V} is negative semi-definite for $\beta = 3$ and V is positive for $\|\underline{x}\| \neq 0$.

Therefore, the system is globally asymptotically stable for $\beta = 3$.

A better result is obtained if d_{23} is not zero. Applying a procedure similar to the above, we have

$$\begin{aligned} V = & x_1^6 + \left[\frac{2d_{23} + d_{23}\beta + 4}{2} \right] x_1^4 + \left[\frac{2d_{23}}{3} \right] x_1^2 + \\ & + d_{23} x_1^3 x_2 + 2x_1^3 x_3 + 2d_{23} x_1 x_2 + \frac{2}{3} d_{23} x_1 x_3 + \\ & + (7/6 d_{23} + 2) x_2^2 + d_{23} x_2 x_3 + x_3^2, \end{aligned}$$

and

$$\dot{V} = -2 \theta x_1^6 + (2\theta + 6 - d_{23}) x_1^3 x_3 + x_3^2 (6 - d_{23}) - \frac{2}{3} d_{23} \theta x_1^4.$$

Let $d_{23} = 2(\theta - 3)$. Then \dot{V} is negative semi-definite and V is positive definite for all $0 < \theta \leq 3$. The system is globally asymptotically stable for this range of θ 's.

(5) Another Third Order System

The describing equations of the system are

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3,$$

$$\dot{x}_3 = -3x_1^2 x_3 - 2x_2 - 6x_1 x_2^2 - x_1^3.$$

In this case the authors constrain \dot{V} such that it is a function of x_2^2 . The form of \dot{V} which must be simplified is

$$\dot{V} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

We start the simplification by letting $d_{31} = 0$. From the curl equations we find that

$$d_{23} = d_{32} = 6x_1^2,$$

$$d_{31} = 12 x_1 x_2.$$

Terms involving $(x_1 x_2)$, $(x_2 x_3)$, and $(x_1 x_3)$ can be eliminated if $d_{11} = 6x_1^4$, $d_{22} = 4$ and $d_{21} = 2x_1^2 + 18x_1^3 x_2$. The term involving $(x_1^3)(x_2^3)$ vanishes if $d_{12} = 36 x_1^3 x_2$. But, it can be shown that the curl equations dictate that d_{12} must actually be $d_{12} = 36x_1^3 x_2 + 6x_1^2$.

Thus, \dot{V} takes the form

$$V = [x_1, x_2, x_3]$$

$$= -6x_1^2 x_2^2.$$

$6x_1^4$	$18x_1^3 x_2 + 2x_1^2$	0	\dot{x}_1
$36x_1^3 x_2 + 6x_1^2$	4	$6x_1^2$	\dot{x}_2
$12x_1 x_2$	$6x_1^2$	2	\dot{x}_3

Integrating ∇V gives

$$V = x_1^6 + 2x_1^3 x_2^2 + 9x_1^4 x_2^2 + 2x_2^2 + 6x_1^2 x_2 x_3 + x_3^2,$$

$$= (x_3 + 3x_1^2 x_2)^2 + 2(x_2 + x_1^3/2)^2 + x_1^6/2.$$

Therefore, \dot{V} is negative semi-definite, V is positive definite and $V \rightarrow \infty$ as $\|x\| \rightarrow \infty$. The system is then globally asymptotically stable.

The next set of examples were obtained from Schultz's paper, reference [5].

In this paper Schultz discusses the generalized Routh-Hurwitz conditions for nonlinear systems of the form

$$x^{(n)} + a_n(x, x^{(1)}, \dots, x^{(n-1)}) x^{(n-1)} + \dots + a_1(x, x^{(1)}, \dots, x^{(n-1)}) x = 0,$$

where the equilibrium state is taken as $x = x^{(1)} = \dots = x^{(n-1)} = 0$. The n^{th} order equation can be written in matrix form:

$$\dot{\underline{x}} = \underline{A}(\underline{x}) \underline{x},$$

$$\underline{A} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_1 & -a_2 & \dots & -a_n \end{bmatrix},$$

$$a_i = a_i(\underline{x}), \quad x_1 = x, \dots, x_n = x^{(n-1)}.$$

The author considers 2nd and 3rd order cases; thus, the gradients of V used in this paper take the forms

$$\nabla V = \begin{array}{|c|c|c|} \hline d_{11}(x_1) & d_{12}(x_1) & x_1 \\ \hline d_{21}(x_1) & d_{22}(x_2) & x_2 \\ \hline \end{array},$$

and

$$\nabla V = \begin{array}{|c|c|c|} \hline d_{11}(x_1) & d_{12}(x_1, x_2) & d_{13}(x_1, x_2) \\ \hline d_{21}(x_1, x_2) & d_{22}(x_2) & d_{23}(x_1, x_2) \\ \hline d_{31}(x_1, x_2) & d_{32}(x_1, x_2) & d_{33}(x_3) \\ \hline \end{array} \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline x_3 \\ \hline \end{array}.$$

(6) General Second Order System

The system is described by

$$\ddot{x} + A(x, \dot{x}) \dot{x} + B_1(x) B_2(\dot{x}) x = 0,$$

or

$$\begin{array}{|c|} \hline \dot{x}_1 \\ \hline \dot{x}_2 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline -B_1 B_2 & -A \\ \hline \end{array} \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \end{array}.$$

The time derivative of V becomes

$$\begin{aligned} \dot{V} &= (\nabla V)_T \dot{x} \\ &= (d_{11} - A d_{21} - d_{22} B_1 B_2) x_1 x_2 + \\ &+ d_{12} x_2^2 - d_{22} A x_2^2 - B_1 B_2 d_{21} x_1^2. \end{aligned}$$

Since d_{12} and d_{21} are functions of x_1 , it is convenient to set them equal to zero in the expression for \dot{V} . Also the $(x_1 x_2)$ term is eliminated in \dot{V} if

$d_{11} = d_{22} B_1 B_2$. To satisfy this equation and the form of ∇V

originally proposed, we choose $d_{11} = B_1(x_1)$ and $d_{22} = 1/B_2(x_2)$.

Thus,

$$\nabla V = \begin{bmatrix} B_1 & 0 \\ 0 & 1/B_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $\partial^2 V / \partial x_1 \partial x_2 = \partial^2 V / \partial x_2 \partial x_1 = 0$. The resulting V and \dot{V} are

$$V = \int_0^{x_1} B_1(x_1) x_1 dx_1 + \int_0^{x_2} \frac{x_2 dx_2}{B_2(x_2)},$$

and

$$\dot{V} = - \left[\frac{A(x_1, x_2)}{B_2(x_2)} \right] x_2^2.$$

For global asymptotic stability of the given system the following conditions are sufficient:

- 1) the non-null solutions of $\dot{V} \equiv 0$ must not be solutions of $\dot{x} = \underline{A} x$;
- 2) $V \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$,
- 3) $A(x_1, x_2) > 0$ for all x_1 and x_2 ,

$$B_1(x_1) > 0 \text{ for all } x_1,$$

$$B_2(x_2) > 0 \text{ for all } x_2.$$

The conditions under (3) are the same as the Routh-Hurwitz conditions for a linear system.

We now consider a third order, nonlinear system defined by

$$\ddot{x} + A\dot{x} + B\dot{x} + Cx = 0,$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -C & -B & -A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The expression for \dot{V} is given by

$$\begin{aligned}\dot{V} &= (\nabla V)_T \dot{\underline{x}} \\ &= (d_{11} - Bd_{31} - cd_{32}) x_1 x_2 + (d_{13} + d_{22} - d_{32}A - 2B) x_2 x_3 + \\ &+ (d_{21} - Ad_{31} - 2C) x_1 x_3 - d_{31} C x_1^2 + \\ &+ (d_{12} - Bd_{32}) x_2^2 - (2A - d_{23}) x_3^2 ,\end{aligned}$$

where $d_{33} = 2$. Below, we consider several special cases of this third order example.

(7) A & B constants; $C = C(x_1)$ (Ingwerson)

The coefficients of $x_1 x_2$, $x_2 x_3$ and x_1^2 in \dot{V} are set equal to zero:

$$d_{11} - B d_{31} - C d_{32} = 0,$$

$$d_{13} + d_{22} - d_{32}A - 2B = 0,$$

$$d_{12} - B d_{32} = 0.$$

Let d_{12} , d_{21} , d_{31} and d_{13} be constants. The curl equations will then impose symmetry; namely, $d_{12} = d_{21}$ and $d_{31} = d_{13}$. For convenience, we

let $d_{32} = d_{23} = A$. Thus, we have

$$d_{32} = A, \quad d_{31} = d_{13} = B, \quad d_{12} = d_{21} = AB,$$

$$d_{22} = A^2 + B, \quad d_{11} = B^2 + AC.$$

The resulting \dot{V} and V are

$$\dot{V} = -BC(x_1) x_1^2 - 2C(x_1) x_1 x_3 - Ax_3^2$$

$$= -\underline{x}_T$$

BC	0	C
0	0	0
C	0	A

$$\underline{x} ,$$

$$\begin{aligned}
 V &= \int_0^{\underline{x}} \underline{\nabla V} \cdot d\underline{x} \\
 &= A \int_0^{x_1} x_1 C(x_1) dx_1 + \left[\frac{B x_1^2}{2} \right] + \left[AB x_1 x_2 \right] + \left[\frac{A^2}{2} + B \right] x_2^2 + \\
 &\quad + B x_1 x_3 + A x_2 x_3 + x_3^2.
 \end{aligned}$$

Thus, the conditions for global asymptotic stability are

$$A > 0$$

$$B > 0$$

$$C(x_1) > 0, \text{ for all } x_1,$$

$$AB - C(x_1) > 0, \text{ for all } x_1.$$

These conditions are analogous to the Routh-Hurwitz conditions for linear systems.

(8) A, C constants; B = B(x₂)

This example is analyzed in the same way as example (7). Thus, we will only give the V-function, V, and the stability requirements. \dot{V} and V are

$$\dot{V} = -2x_2^2 \left[A B(x_2) - C \right],$$

and

$$\begin{aligned}
 V &= ACx_1^2 + 2Cx_1x_2 + A^2 x_2^2 + 2Ax_2x_3 + x_3^2 + \\
 &\quad + 2 \int_0^{x_2} B(x_2) x_2 dx_2.
 \end{aligned}$$

The stability requirements are again the generalized Routh-Hurwitz conditions:

$$A > 0,$$

$$B(x_2) > 0, \text{ for all } x_2,$$

$$C > 0,$$

$$AB(x_2) - C > 0, \text{ for all } x_2.$$

- (9) B and C are constants; A = A(x₂) (LaSalle)

This example is also very similar to (7) and (8) and thus, only V, and \dot{V} , and the stability requirements are given. V and \dot{V} are

$$V = \frac{C^2}{B} x_1^2 + 2Cx_1x_2 + Bx_2^2 + \left[\frac{2C}{B}\right] x_2x_3 + x_3^2 + \\ + \frac{2C}{B} \int_0^{x_2} A(x_2) x_2 dx_2,$$

and

$$\dot{V} = - 2x_3^2/B (A(x_2) B - C) .$$

Once again the stability conditions are the generalized Routh-Hurwitz conditions:

$$A(x_2) > 0, \text{ for all } x_2,$$

$$B > 0,$$

$$C > 0,$$

$$A(x_2) B - C > 0, \text{ for all } x_2.$$

- (10) A = constant, B = B(x), C = C(x) (Barbashin)

One of the most general third order cases considered by Schultz was Barbashin's problem:

$$\ddot{x} + A\ddot{x} + B(\dot{x}) \dot{x} + C(x) x = 0.$$

By a procedure similar to example (7) we obtain the following V and \dot{V} :

$$V = 2A \int_0^{x_1} C(x_1) x_1 dx_1 + 2C(x_1) x_1x_2 + A^2x_2^2 + \\ + 2 \int_0^{x_2} B(x_2) x_2 dx_2 + 2Ax_2x_3 + x_3^2,$$

and

$$\dot{V} = - 2x_2^2 \left[AB(x_2) - C(x_1) \right] + 2x_2^2 \left[\frac{dC(x_1)}{dx_1} \right] x_1$$

The resulting stability conditions are

$$(1) \quad A > 0,$$

$$B(x_2) > 0, \text{ for all } x_2,$$

$$C(x_1) > 0, \text{ for all } x_1,$$

$$AB(x_2) - c(x_1) > 0, \text{ for all } x_1 \text{ and } x_2,$$

$$(2) \quad \frac{dC(x_1)}{dx_1} x_1 \leq 0.$$

The conditions under (1) are the generalized Routh-Hurwitz conditions, and

(2) represents a saturating type of nonlinearity found in control systems.

(11) Third Order System

The following equations describe a motor compensated with tachometer feedbacks:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = - \left(\beta g(x_1) x_1 - \left[N + \frac{df(x_1)}{dx_1} \right] x_2 - Mx_3 \right).$$

where

$$f(x_1) = x_1 g(x_1).$$

This stability problem is more difficult than the previous problems. We must consider the d 's to be made up of a constant term and a variable term.

We assume a_{33} equals 2. Thus, \dot{V} becomes

$$\begin{aligned} \dot{V} = & x_1 x_2 (d_{11} - d_{31} N - d_{31} f_1' - d_{32} \beta g_1) + \\ & + x_2 x_3 (d_{13} + d_{22} - d_{32} M - 2N - 2f_1') + \\ & + x_1 x_3 (d_{21} - d_{31} M - 2\beta g_1) - d_{31} \beta g_1 x_1^2 + \\ & + x_2^2 (d_{12} - d_{32} N - d_{32} f_1') + x_3^2 (d_{23} - 2M), \end{aligned}$$

where $f_1' = df(x_1)/dx_1$ and $g_1 = g(x_1)$. We now simplify \dot{V} .

First, cancel-out the β -dependent term in the coefficient of x_1x_2 by a proper choice of d_{11} . This leads to the result that \dot{V} is easily constrained in terms of x_1 and x_3 ; therefore, make the x_2 terms vanish. Then, the curl equations are applied. The results of these manipulations are

$$\begin{aligned} d_{23} &= d_{32} = 2(M - \beta) , \\ d_{22} &= d_{23} (M - N/M) + 2N , \\ d_{21} &= d_{23} (N + g_1) , \\ d_{12} &= d_{23} (N + f_1') , \\ d_{31} &= d_{23} N/M + 2g_1 , \\ d_{13} &= d_{23} N/M + 2f_1' , \\ d_{11} &= d_{31} (N + f_1') + d_{23} \beta g_1 ; \end{aligned}$$

therefore, \dot{V} becomes

$$\begin{aligned} \dot{V} &= \left\{ -2\beta g_1^2 x_1^2 - x_1x_3g_1 \left[2\beta + 2M - d_{23} \right] - x_3^2 \left[2M - d_{23} \right] \right\} + \\ &\quad + \left[-\frac{d_{23}N}{M} \right] \beta g_1 x_1^2 . \end{aligned}$$

If the following substitutions are made in the $\{ \}$ - term

$$\begin{aligned} d_{23} &= 2(M - \beta) , \\ x_1g_1 &= Z_1 , \\ x_3 &= Z_3 , \end{aligned}$$

then \dot{V} becomes

$$V = -2\beta (Z_1 + Z_3)^2 - \left[\frac{2(M-\beta)\beta N}{M} \right] g_1 x_1^2 .$$

Thus, \dot{V} is negative semi-definite if we demand that

$$\begin{aligned} N &> 0, \quad M > 0 , \\ 0 &< \beta \leq M , \\ g(x_1) &> 0, \text{ for all } x_1 . \end{aligned}$$

Also, we assume that $\dot{V} \equiv 0$ has no non-null solutions. By line integration, V is determined to be

$$\begin{aligned}
 V = & (2N + d_{23} \beta) \int_0^{x_1} g(\gamma_1) \gamma_1 d\gamma_1 + \left[\frac{d_{23} N}{M} \right] \int_0^{x_1} f'(\gamma_1) \gamma_1 d\gamma_1 + \\
 & + \left[g(x_1) x_1 \right]^2 + \left[\frac{d_{23} N^2 x_1^2}{2M} \right] + d_{23} N x_1 x_2 + d_{23} g(x_1) x_1 x_2 + \\
 & + \left[d_{23} M + 2N - \frac{d_{23} N}{M} \right] x_1^2 + \left[\frac{d_{23} N}{M} \right] x_1 x_3 + \\
 & + 2 g(x_1) x_1 x_3 + d_{23} x_2 x_3 + x_3^2 .
 \end{aligned}$$

The conclusions obtained from this very complicated V -function are summarized by Schultz. The class of functions for which his conclusions are valid is defined as the set of continuous nonlinear functions with positive slopes and which lie in the first and third quadrants of the $x_1 y$ - plane. Thus, V is shown to be positive definite by Ingwerson's method. Therefore, our system is globally asymptotically stable whenever $V \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

From Geiss' Report, [7], we consider Duffing's equation as analyzed by the variable gradient method.

(12) Duffing's Equation

The defining equations are

$$\begin{aligned}
 \dot{x}_1 &= x_2, \\
 \dot{x}_2 &= -x_1 - b x_1^3, \quad b > 0.
 \end{aligned}$$

The choice for ∇V is given by

$a_{11} + d_{11}(x_1)$	$a_{12} + d_{12}(x_1)$	x_1
$a_{21} + d_{21}(x_1)$	2	x_2

$\nabla V =$

where a_{11} , a_{12} and a_{21} are constants. From ∇V and the defining equations, \dot{V} becomes

$$\begin{aligned}\dot{V} &= (\nabla V)_T \dot{\mathbf{x}} = (a_{11} + d_{11}) x_1 x_2 + (a_{12} + d_{12}) x_2^2 + \\ &\quad - (a_{21} + d_{21}) x_1^2 - 2x_1 x_2 + \\ &\quad - b (a_{21} + d_{21}) x_1^4 - 2b x_1^3 x_2.\end{aligned}$$

Thus, \dot{V} is negative definite if $a_{11} = 2$, $d_{11} = 2b x_1^2$, $d_{12} = d_{21} = 0$, and $a_{12} < 0$. The curl equation gives

$$\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial^2 V}{\partial x_2 \partial x_1} = a_{12} = a_{21} < 0.$$

Let $a_{12} = a_{21} = -\gamma$, where $\gamma > 0$. Therefore,

V and \dot{V} become

$$V = \left[\frac{b x_1^4}{2} \right] + x_1^2 - \gamma x_1 x_2 + x_2^2,$$

and

$$\dot{V} = -\gamma \left[x_2^2 - x_1^2 - b x_1^4 \right].$$

V is positive definite if $0 \leq \gamma < 2$. \dot{V} is indefinite unless $\gamma = 0$.

When $\gamma = 0$, V is positive definite and \dot{V} is identically zero. In fact, since $\dot{V} \equiv 0$, $V = \text{constant}$ is a trajectory of the system. The system is stable in the neighborhood of the null solution, but not asymptotically stable.

We now consider some examples analyzed by Puri's "shorthand" method for the variable gradient technique. One example will be a generalization of Barbashin's problem, [8]. The other examples will be the same ones as considered by Schultz and Gibson in [1].

(13) Second Order System

The system is described by

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1^3 - x_2,\end{aligned}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -x_1^2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We assume a symmetric form for the square matrix in ∇V :

$$\nabla V = \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{G} \underline{x}.$$

The time derivative of V is

$$\begin{aligned} \dot{V} &= (\nabla V)_T \dot{\underline{x}} = \underline{x}_T (G_T \underline{A}) \underline{x} = \underline{x}_T \underline{T} \underline{x}, \\ &= \underline{x}_T \begin{bmatrix} -G_{12}x_1^2 & G_{11} - G_{12} \\ -x_1^2 & G_{12} - 1 \end{bmatrix} \underline{x}, \\ &= \underline{x}_T \begin{bmatrix} -G_{12}x_1^2 & G_{11} - G_{12} - x_1^2 \\ 0 & G_{12} - 1 \end{bmatrix} \underline{x}. \end{aligned}$$

Let $G_{12} = 0$ and $G_{11} = x_1^2$. Then $\dot{V} = -x_2^2$. The gradient is

$$\nabla V = \begin{bmatrix} 2x_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1^2 \\ x_2 \end{bmatrix},$$

where $\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial^2 V}{\partial x_2 \partial x_1} = 0$. By line integration

$$V = x_1^4/4 + x_2^2/2.$$

Thus, V is positive definite; \dot{V} is negative semi-definite; $V \rightarrow \infty$ as

$\|\underline{x}\| \rightarrow \infty$; and $\dot{V} \equiv 0$ is satisfied only by the null solution.

Therefore, the system is globally asymptotically stable.

(14) A General Second Order System

The system is described by

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -b(x_1) x_1 - a(x_1, x_2) x_2,$$

or

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \underline{x}.$$

The choice for ∇V is the same as in the previous example; thus V becomes

$$\begin{aligned} \dot{V} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ &= \underline{x}_T \begin{bmatrix} -bG_{12} & G_{11} - aG_{12} \\ -b & G_{12} - a \end{bmatrix} \underline{x}, \\ &= \underline{x}_T \begin{bmatrix} -bG_{12} & G_{11} - aG_{12} - b \\ 0 & G_{12} - a \end{bmatrix} \underline{x}, \end{aligned}$$

Let $G_{12} = 0$ and $G_{11} = b$. Therefore, ∇V becomes:

$$\nabla V = \begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $\partial V / \partial x_1 = b(x_1) x_1$ and $\partial V / \partial x_2 = x_2$. The curl equation is easily checked:

$$\frac{\partial^2 V}{\partial x_1 \partial x_2} = \frac{\partial^2 V}{\partial x_2 \partial x_1} = 0.$$

By line integration of ∇V we get

$$V = \int_0^{x_1} x_1 b(x_1) dx_1 + x_2^2 / 2,$$

where $\dot{V} = -a x_2^2$. The system is globally asymptotically stable if

$$b(x_1) > 0 \text{ for all } x_1,$$

$$\int_0^{x_1} b(x_1) x_1 dx_1 \longrightarrow \infty \text{ as } x_1 \longrightarrow \infty,$$

$$a(x_1, x_2) > 0 \text{ for all } x_1 \text{ and } x_2.$$

(15) A General Third Order System

This system comes from Puri's work in reference [8]. The equation is

$$\ddot{x} + f_3(\ddot{x}) + f_2(\dot{x}) + f_1(x) = 0.$$

The matrix formulation, where $x_1 = x$, $x_2 = \dot{x}$ and $x_3 = \ddot{x}$, is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -f_1/x_1 & -f_2/x_2 & -f_3/x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The equilibrium solution is assumed to occur at $\underline{x} = \underline{0}$. The form of ∇V is assumed to be

$$\nabla V = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{12} & G_{22} & G_{23} \\ G_{13} & G_{23} & 1 \end{bmatrix} \underline{x} = \underline{G} \underline{x}.$$

Since in \underline{G} there exists five unknown functions of \underline{x} , we need five equations involving the G_{ij} 's. These relationships are obtained from \dot{V} as follows:

$$\dot{V} = (\nabla V)_T \dot{\underline{x}} = \underline{x}_T \underline{G}_T \underline{A} \underline{x},$$

$$= \underline{x}_T \begin{bmatrix} -G_{13} f_1/x_1 & G_{11} - G_{13} f_2/x_2 & G_{12} - G_{13} f_3/x_3 \\ -G_{23} f_1/x_1 & G_{12} - G_{23} f_2/x_2 & G_{22} - G_{23} f_3/x_3 \\ -f_1/x_1 & G_{13} - f_2/x_2 & G_{23} - f_3/x_3 \end{bmatrix} \underline{x},$$

$= \underline{x}$	$-G_{13} f_1/x_1$	$G_{11} - G_{13} f_2/x_2 - G_{23} f_1/x_1$	$G_{12} - G_{13} f_3/x_3 - f_1/x_1$
	0	$G_{12} - G_{23} f_2/x_2$	$G_{22} - G_{23} f_3/x_3 + G_{13} - f_2/x_2$
	0	0	$G_{23} - f_3/x_3$

Let all the elements in the above triangular matrix be zero except the second row, second column element. Thus, we get the following results:

$$\begin{aligned} G_{13} &= 0, \quad G_{12} = f_1(x_1)/x_1, \\ G_{23} &= f_3(x_3)/x_3, \quad G_{11} = f_1(x_1) f_3(x_3)/x_1 x_3, \\ G_{22} &= f_2(x_2)/x_2 + \left[f_3(x_3)/x_3 \right]^2. \end{aligned}$$

Since $\nabla V = \underline{G} \underline{x}$, we have

$$\begin{aligned} \frac{\partial V}{\partial x_1} &= f_1 f_3/x_3 + f_1 x_2/x_1, \\ \frac{\partial V}{\partial x_2} &= f_1 + f_2 + x_2 f_3^2/x_3 + f_3 x_3/x_3, \\ \frac{\partial V}{\partial x_3} &= x_2 f_3/x_3 + x_3. \end{aligned}$$

We now see one of the disadvantages of assuming a symmetric \underline{G} matrix when we attempt to check the curl equations. The curl equations produce the following:

$$\begin{aligned} \frac{\partial^2 V}{\partial x_1 \partial x_2} &= \frac{\partial^2 V}{\partial x_2 \partial x_1} \quad \text{gives} \quad f_1/x_1 = \frac{df_1}{dx_1}, \\ \frac{\partial^2 V}{\partial x_1 \partial x_3} &= \frac{\partial^2 V}{\partial x_3 \partial x_1} \quad \text{gives} \quad \left[f_1 \frac{d(f_3/x_3)}{dx_3} \right] = 0, \\ \frac{\partial^2 V}{\partial x_2 \partial x_3} &= \frac{\partial^2 V}{\partial x_3 \partial x_2} \quad \text{gives} \quad \frac{d(f_3/x_3)^2}{dx_3} + \frac{d(f_3/x_3^3)}{dx_3} = f_3/x_3. \end{aligned}$$

One way in which these curl equations could be satisfied is to replace f_3/x_3 by a constant; namely, $\alpha = \lim_{x_3 \rightarrow 0} (f_3/x_3)$, and to let G_{12} equal df_1/dx_1 . Our new \underline{G} is no longer symmetric, but is

$$\underline{G} = \begin{array}{|c|c|c|} \hline \alpha f_1/x_1 & f_1' & 0 \\ \hline f_1/x_1 & \alpha^2 + f_2/x_2 & \alpha \\ \hline 0 & \alpha & 1 \\ \hline \end{array},$$

resulting in the following expressions for the components of ∇V ,

$$\frac{\partial V}{\partial x_1} = [\alpha f_1 + f_1' x_2],$$

$$\frac{\partial V}{\partial x_2} = [f_1 + f_2 + \alpha^2 x_2 + \alpha x_3],$$

$$\frac{\partial V}{\partial x_3} = [\alpha x_2 + x_3],$$

and V and \dot{V} become

$$\begin{aligned} V &= \int_{(0,0,0)}^{(x_1,0,0)} \left[\frac{\partial V}{\partial x_1} \right] dx_1 + \int_{(x_1,0,0)}^{(x_1,x_2,0)} \left[\frac{\partial V}{\partial x_2} \right] dx_2 + \int_{(x_1,x_2,0)}^{(x_1,x_2,x_3)} \left[\frac{\partial V}{\partial x_3} \right] dx_3, \\ &= \alpha \int_0^{x_1} f_1(x_1) dx_1 + x_2 f_1(x_1) + \int_0^{x_2} f_2(x_2) dx_2 + \frac{\alpha^2}{2} \left(x_2^2 \right) + \\ &\quad + \alpha x_2 x_3 + x_3^2/2, \\ &= \frac{1}{2} (\alpha x_2 + x_3)^2 + f_1(x_1) x_2 + \alpha \int_0^{x_1} f_1(x_1) dx_1 + \int_0^{x_2} f_2(x_2) dx_2, \end{aligned}$$

and

$$\begin{aligned} \dot{V} &= (\nabla V)_T \dot{\underline{x}} \\ &= - \left\{ \left[\alpha f_2/x_2 - f_1' \right] x_2^2 + 2 \left[\frac{\alpha}{2} (f_3/x_3 - \alpha) \right] x_2 x_3 + \left[f_3/x_3 - \alpha \right] x_3^2 \right\} \\ &= - \begin{bmatrix} x_2 & x_3 \end{bmatrix} \begin{bmatrix} \alpha f_2/x_2 - f_1' & \frac{\alpha}{2} (f_3/x_3 - \alpha) \\ \frac{\alpha}{2} (f_3/x_3 - \alpha) & f_3/x_3 - \alpha \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \end{aligned}$$

Therefore, a set of sufficient conditions for a global asymptotically stable system is as follows:

$$\alpha > 0, \quad f_1(x_1) x_1 > 0, \quad f(x_2) x_2 > 0,$$

$$\alpha f_2(x_2)/x_2 - \frac{d f_1(x_1)}{dx_1} > 0,$$

$$4 \left[\alpha f_2(x_2) / x_2 - \frac{df_1(x_1)}{dx_1} \right] \geq \alpha^2 \left[\frac{f_3(x_3)}{x_3} - \alpha \right] \geq 0,$$

$$1/2 (\alpha x_2 + x_3)^2 + \alpha \int_0^{x_1} f_1(x_1) dx_1 + \int_0^{x_2} f_2(x_2) dx_2 \geq \left| f_1(x_1) x_2 \right|,$$

and

f_1, f_2, f_3 are such that $V \rightarrow \infty$ as $\| \underline{x} \| \rightarrow \infty$.

Barabashin's problem is a special case of the above if we let $f_3/x_3 = \alpha$ in the equation describing the system. The corresponding V and \dot{V} are

$$\dot{V} = - \left[\alpha f_2/x_2 - f_1' \right] x^2,$$

$$V = 1/2 (\alpha x_2 + x_3)^2 + f_1(x_1) x_2 + \alpha \int_0^{x_1} f_1(x_1) dx_1 + \int_0^{x_2} f_2(x_2) dx_2.$$

The stability requirements reduce to

$$\alpha > 0, f_1(x_1) x_1 > 0, f_2(x_2) x_2 > 0,$$

$$\alpha f_2(x_2)/x_2 - f_1'(x_1) \geq 0,$$

$$1/2 (\alpha x_2 + x_3)^2 + \alpha \int_0^{x_1} f_1(x_1) dx_1 + \int_0^{x_2} f_2(x_2) dx_2 \geq |f_1(x_1)x_2|.$$

Also, the third order linear system is a special case of the above problem.

Let $f_3 = a_3 x_3 = \alpha x_3$, $f_2 = a_2 x_2$ and $f_1 = a_1 x_1$, where the a 's are all constants. Thus, V and \dot{V} are

$$\dot{V} = - \begin{bmatrix} a_2 & a_3 & -a_1 \end{bmatrix} x_3^2,$$

$$V = 1/2 (a_3 x_2 + x_3)^2 + a_1 x_1 x_2 + \left[\frac{a_1 a_3}{2} \right] x_1^2 + \left[\frac{a_2}{2} \right] x_2^2.$$

The stability conditions are

$$a_1 > 0, a_2 > 0, a_3 > 0,$$

$$a_2 a_3 - a_1 > 0,$$

which correspond to the Routh-Hurwitz conditions.

The next set of examples comes from Ku and Puri's work which is reported in references [9] and [10].

(16) Simanov's Third-Order System, [10]

The nonlinear differential equation is

$$\ddot{x} + f(x, \dot{x}) \ddot{x} + b \dot{x} + cx = 0,$$

where b and c are constants. For the state variables $x_1 = x$, $x_2 = \dot{x}$ and

$x_3 = \ddot{x}$, we have

$$\dot{\underline{x}} = \underline{A}(\underline{x}) \underline{x} =$$

0	1	0
0	0	1
-c	-b	-f(x ₁ , x ₂)

\underline{x} .

The form of the \underline{S} matrix for this system with one nonlinearity is

$$2 \underline{S} = \begin{array}{|c|c|c|} \hline K_{11} & K_{12} & K_{13} \\ \hline K_{12} & K_{22} + Y(x_1, x_2)/x_2^2 & K_{23} \\ \hline K_{13} & K_{23} & K_{33} \\ \hline \end{array},$$

where the K_{ij} 's are constants. The candidate for V is $V = \underline{x}_T \underline{S} \underline{x}$. The gradient of V becomes

$$\underline{\nabla V} = \underline{B} \underline{x} = \begin{array}{|c|c|c|} \hline K_{11} + \frac{1}{2x_1} \frac{\partial Y}{\partial x_1} & K_{12} & K_{13} \\ \hline K_{12} & K_{22} + \frac{1}{2x_2} \frac{\partial Y}{\partial x_2} & K_{23} \\ \hline K_{13} & K_{23} & K_{33} \\ \hline \end{array} \underline{x}.$$

Thus, the time derivative of V yields

$$\dot{V} = \underline{x}_T \underline{B}_T \underline{A} \underline{x} = \underline{x}_T \underline{T} \underline{x} = \begin{array}{|c|c|c|} \hline -CK_{13} & K_{11} + \frac{1}{2x_1} \frac{\partial Y}{\partial x_1} - bK_{13} & K_{12} - fK_{13} \\ \hline -CK_{23} & K_{12} - bK_{23} & K_{22} + \frac{1}{2x_2} \frac{\partial Y}{\partial x_2} - fK_{23} \\ \hline -CK_{33} & K_{13} - bK_{33} & K_{23} - fK_{33} \\ \hline \end{array} \underline{x}.$$

For $T_{13} + T_{31} = 0$, $K_{13} = 0$ and $K_{12} = cK_{33}$. For the constant parts of $T_{12} + T_{21}$ to be zero we require that $K_{13} = 0$ and $K_{11} = cK_{23}$. Let T_{22} be zero; thus $bK_{23} = K_{12}$. Let the constant parts of $T_{23} + T_{32}$ be zero; therefore $K_{13} = 0$ and $K_{22} = bK_{33}$. Choose K_{33} to be b^2 . The remaining constants are $K_{22} = b^3$, $K_{23} = bc$, $K_{12} = b^2c$, $K_{11} = bc^2$. Let the variable parts in T_{23} be zero, thus

$$\partial Y / \partial x_2 = 2bc f(x_1, x_2) x_2,$$

$$Y = 2bc \int_0^{x_2} f(x_1, x_2) x_2 dx_2.$$

Therefore, it follows that

$$\frac{\partial Y}{\partial x_1} = 2 bc \int_0^{x_2} \frac{\frac{\partial f(x_1, x_2)}{\partial x_1}}{x_2} dx_2.$$

The resulting matrix \underline{T} in \dot{V} becomes

$$\underline{T} = \begin{array}{|c|c|c|} \hline 0 & bc/x_1 \int_0^{x_2} \left[\frac{\partial f}{\partial x_1} \right] x_2 dx_2 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & -b^2 (f - c/b) \\ \hline \end{array}$$

The final expression for \dot{V} is

$$\dot{V} = -b^2 \left[f(x_1, x_2) - c/b \right] x_2^2 + bcx_2 \int_0^{x_2} \left[\frac{\partial f}{\partial x_1} \right] x_2 dx_2.$$

Substituting the K_{ij} 's and Y into \underline{S} we get our V - function:

$$2V = b (cx_1 + bx_2)^2 + (cx_2 + bx_3)^2 + J(x_1, x_2),$$

where

$$J(x_1, x_2) = Y(x_1, x_2) - c^2 x_2^2.$$

The system is asymptotically stability if we require

$$2b \int_0^{x_2} f(x_1, x_2) x_2 dx_2 > cx_2^2, \quad b > 0,$$

$$f(x_1, x_2) > c/b,$$

$$\left[b f(x_1, x_2) - c \right] x_2^2 > cx_2 \int_0^{x_2} \left[\frac{\partial f}{\partial x_1} \right] x_2 dx_2,$$

$$J(x_1, x_2) \longrightarrow \infty \text{ as } \|x\| \longrightarrow \infty,$$

and $\dot{V} \equiv 0$ only for the null solution.

(17) Cartwright's Fourth Order Example [9]

The system is described by

$$\ddot{\ddot{x}} + a_4 \ddot{x} + a_3 \dot{x} + a_4 \dot{x} + f(x) = 0 ,$$

or

$$\dot{\underline{x}} = \begin{array}{c|ccc} & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline -f(x_1) / x_1 & -a_2 & -a_3 & -a_4 \end{array} \underline{x} .$$

The \underline{S} matrix which the authors choose is

$$\underline{S} = \begin{array}{c|ccc} K_{11} + \frac{Y(x_1)}{x_1^2} & K_{12} + \frac{f_{12}(x_1)}{2x_1} & K_{13} + \frac{f_{13}(x_1)}{x_1} & K_{14} \\ \hline K_{12} + \frac{f_{12}(x_1)}{2x_1} & K_{22} - \frac{1}{2} f'(x_1) & K_{23} & K_{24} \\ K_{13} + \frac{f_{13}(x_1)}{2x_1} & K_{23} & K_{33} & K_{34} \\ \hline K_{14} & K_{24} & K_{34} & K_{44} \end{array} ,$$

where K_{ij} 's are constants. The \underline{T} matrix is formed as before and the results are

$$K_{11} = 0, \quad K_{22} = 1/2 (\Delta_3 a_3 + a_2 a_4), \quad \Delta_3 = a_3 - a_2/a_4,$$

$$K_{33} = 1/2 (a_4^2 + a_2/a_4), \quad K_{44} = 1/2 ,$$

$$K_{12} = 0, \quad K_{13} = K_{14} = 0, \quad K_{24} = 1/2 \Delta_3 ,$$

$$K_{23} = 1/2 a_3 a_4, \quad K_{34} = 1/2 a_4$$

$$f_{12} = a_4 f, \quad f_{13} = f, \quad \frac{dY}{dx_1} = \Delta_3 f .$$

Thus, V becomes

$$\begin{aligned} 2V = & (x_4 + a_4 x_3 + \Delta_3 x_2)^2 + \frac{a_2}{a_4} \left(x_3 + a_4 x_2 + \frac{a_4}{a_2} f(x_1) \right)^2 + \\ & + \left[\frac{a_2}{a_4} \Delta_3 - f' \right] x_2^2 + 2 \Delta_3 \int_0^{x_1} f(x_1) dx_1 - \frac{a_4}{a_2} f^2(x_1) . \end{aligned}$$

The time derivative of V is

$$\dot{V} = - \left[(a_3 - a_2/a_4) a_2 - a_4 f' + f''/2 x_2 \right] x_2^2 .$$

We have an asymptotically stable system if we require the following:

$$a_2 > 0, a_3 > 0, a_4 > 0, \Delta_3 > 0,$$

$$\left[\Delta_3 a_2 - a_4 f' + (f''/2) x_2 \right] > 0,$$

$$a_2/a_4 \Delta_3 - f' > 0,$$

$$2 \Delta_3 \int_0^{x_1} f(x_1) dx_1 - a_4/a_2 f^2(x_1) > 0 ,$$

and (f) is such that $V \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$.

(18) Ku's Fourth Order Example [10]

The nonlinear differential equation is given by

$$\ddot{\ddot{x}} + a \ddot{\ddot{x}} + f(x, \dot{x}) \ddot{x} + c\dot{x} + dx = 0,$$

or in state variable notation:

$$\dot{\underline{x}} = \underline{A} \underline{x} + \underline{B} \underline{u},$$

0	1	0	0
0	0	1	0
0	0	0	1
-d	-c	-f(x ₁ , x ₂)	-a

where $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, $x_4 = \ddot{\ddot{x}}$. The \underline{S} matrix in $V = \underline{x}^T \underline{S} \underline{x}$ is

chosen such that the \underline{B} matrix in $\nabla V = \underline{B} \underline{x}$ has the following form:

$$\underline{B} =$$

$K_{11} + \frac{1}{2x_1} \frac{\partial Y}{\partial x_1}$	K_{12}	$K_{13} + \frac{x_3}{2} \frac{\partial Y}{\partial x_1}$	K_{14}
K_{12}	$K_{22} + \frac{1}{2x_2} \frac{\partial Y}{\partial x_2}$	$K_{23} + \frac{x_3}{2} \frac{\partial Y}{\partial x_2}$	K_{24}
K_{13}	K_{23}	$K_{33} + Y$	K_{34}
K_{14}	K_{24}	K_{34}	K_{44}

where $Y = Y(x_1, x_2)$, $\gamma = \gamma(x_1, x_2)$

and the K_{ij} 's are constants. The time derivative of V is given by

$\dot{V} = \underline{x}_T \underline{B}_T \underline{A} \underline{x} = \underline{x}_T \underline{T}_2 \underline{x}$, where the elements of \underline{T}_2 are chosen such that \underline{T}_2 is negative semi-definite. The resulting relationships for the K_{ij} 's and the unknown functions are:

$$K_{44} = 1, K_{34} = a, K_{12} = ad, K_{14} = 0,$$

$$K_{13} = d, K_{24} = ad/c, K_{11} = ad^2/c,$$

$$K_{23} = c + a^2d/c, K_{22} = ac - d,$$

$$K_{33} = a^2 - ad/c, \gamma = f(x_1, x_2),$$

$$\frac{\partial Y}{\partial x_2} = 2 ad/c f(x_1, x_2) x_2,$$

$$Y = 2 ad/c \int_0^{x_2} f(x_1, x_2) x_2 dx_2.$$

A simplified equivalent \underline{T} matrix is

$$\underline{T} =$$

0	$\frac{ad}{cx_1} \int_0^{x_2} \frac{\partial f}{\partial x_2} x_2 dx_2$	0	0
0	0	0	0
0	$\frac{x_3}{2} \cdot \frac{\partial f}{\partial x_1}$	$-af + c + \frac{a^2d}{c} + \frac{x_3}{2} \frac{\partial f}{\partial x_2}$	0
0	0	0	0

and the corresponding time derivative is

$$\begin{aligned} \dot{V} = \underline{x}_T \underline{T} \underline{x} = & - \left[af - c - a^2d/c - 1/2 \left[\frac{\partial f}{\partial x_1} x_2 + \frac{\partial f}{\partial x_2} x_3 \right] \right] x_3^2 + \\ & + ad/c x_2 \int_0^{x_2} \left[\frac{\partial f}{\partial x_1} \right] x_2 dx_2. \end{aligned}$$

The V-function is given as

$$2V = c/a \left[\frac{ad}{c} x_1 + ax_2 + x_3 \right]^2 + \left[\frac{ad}{c} x_2 + ax_3 + x_4 \right]^2 + \left[f - c/a - \frac{ad}{c} \right] x_3^2 + \left[2aY - d/c (c + \frac{a^2d}{c}) x_2^2 \right].$$

Asymptotic stability of the system will be guaranteed if

$$af - c - a^2d/c - 1/2 \left[\left(\frac{\partial f}{\partial x_1} \right) x_2 + \left(\frac{\partial f}{\partial x_2} \right) x_3 \right] \geq \epsilon > 0,$$

$$\epsilon \equiv ad/c (x_2/x_3)^2 \int_0^{x_2} \left(\frac{\partial f}{\partial x_1} \right) x_2 dx_2,$$

$$f - c/a - ad/c > 0,$$

$$Y - d/c (c + a^2d/c) x_2^2 > 0$$

and f is such that $V \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

The next several examples are from the Ph. D. thesis of Mekel, [1].

There are second order, third order and fourth order examples. Some of the systems are the same as considered before but Mekel's Liapunov functions and stability conditions are different. Because the method of Ku, Puri and Mekel has been discussed in detail in the text of this section as well as in the examples, only the salient points of the following examples will be given.

(19) $\ddot{x} + a_2 \dot{x} + f(x) = 0$, $a_2 = \text{constant}$

The matrix form of the equation is

0	1
$-\frac{f(x_1)}{x_1}$	$-a_2$

 $\dot{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad x = x_1, \quad \dot{x} = x_2.$

The Liapunov function and the corresponding time derivative are

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{array}{|c|c|} \hline 2F(x_1)/x_1^2 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \underline{x} ,$$

and

$$\dot{V} = \underline{x}_T \underline{T} \underline{x} = \underline{x}_T \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & -a_2 \\ \hline \end{array} \underline{x} ,$$

where $F(x_1) = \int_0^{x_1} f(x_1) dx_1$. The resulting conditions for asymptotic stability are

$$a_2 > 0, \quad x_1 f(x_1) > 0 \text{ if } x_1 \neq 0,$$

$$F(x_1) \rightarrow \infty \text{ if } |x_1| \rightarrow \infty.$$

$$(20) \quad \underline{\ddot{x}} + g(x) \underline{\dot{x}} + a_1 \underline{x} = 0, \quad \underline{a_1} \equiv \text{constant}$$

The matrix form of the equation is

$$\underline{\dot{x}} = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline -a_1 & -g(x_1) \\ \hline \end{array} \underline{x} ; \quad x = x_1, \quad \dot{x} = x_2.$$

The Liapunov function and the time derivative are

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{array}{|c|c|} \hline a_1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \underline{x} ,$$

and

$$\dot{V} = \underline{x}_T \underline{T} \underline{x} = 2 \underline{x}_T \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & -g(x_1) \\ \hline \end{array} \underline{x} ,$$

The resulting conditions for asymptotic stability are $a_1 > 0$ and $g(x_1) > 0$ for all x_1 .

$$(21) \quad \ddot{x} + g(x) \dot{x} + f(x) = 0$$

The matrix equation is

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{f(x_1)}{x_1} & -g(x_1) \end{bmatrix} \underline{x} ; \quad x = x_1, \quad \dot{x} = x_2 .$$

The Liapunov function and the time derivative are

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{bmatrix} \frac{2F(x_1)}{x_1^2} & 0 \\ 0 & 1 \end{bmatrix} \underline{x} ,$$

$$\dot{V} = \underline{x}_T \underline{T} \underline{x} = 2\underline{x}_T \begin{bmatrix} 0 & 0 \\ 0 & -g(x_1) \end{bmatrix} \underline{x} ,$$

where $F(x_1) = \int_0^{x_1} f(x_1) dx_1$. The resulting conditions for asymptotic stability are

$$g(x_1) > 0 ,$$

$$x_1 f(x_1) > 0, \quad x_1 \neq 0 ,$$

$$F(x_1) \longrightarrow \infty \quad \text{as} \quad |x_1| \longrightarrow \infty .$$

$$(22) \quad \underbrace{\ddot{\ddot{x}} + a_4 \ddot{\ddot{x}} + \phi(x, \dot{x}) \ddot{\ddot{x}} + g(\dot{x}) \ddot{\ddot{x}} + f(x)} = 0$$

This fourth order system can be expressed in matrix form as

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -f(x_1)/x_1 & -g(x_2)/x_2 & -\phi(x_1, x_2) & -a_4 \end{bmatrix} \underline{x} ,$$

where $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, $x_4 = \dddot{x}$ and a_4 is a constant. In the following \underline{S} matrix the K_{ij} 's are constants, $Y_1 = Y_1(x_1)$, $Y_2 = Y_2(x_2)$, $f_1 = f_1(x_1)$, $f_2 = f_2(x_2)$, $g_1 = g_1(x_2)$, $\phi_1 = \phi_1(x_1, x_2)$ and $Y = Y(x_1, x_2)$:

$$\underline{S} = \begin{array}{|c|c|c|c|} \hline K_{11} + \frac{Y_1 + Y}{x_1^2} & K_{12} + f_1/x_1 & K_{13} + f_2/x_1 & K_{14} \\ \hline K_{12} + f_1/x_1 & K_{22} + \frac{Y_2 + Y}{x_2^2} & K_{23} + g_1/x_2 & K_{24} \\ \hline K_{13} + f_2/x_1 & K_{23} + g_1/x_2 & K_{33} + \phi_1 & K_{34} \\ \hline K_{14} & K_{24} & K_{34} & 1 \\ \hline \end{array}$$

The time derivative $\dot{\underline{V}} = \underline{x}_T^T \underline{T} \underline{x}$ is now formed and the unknown constants and functions in \underline{S} are determined by making \underline{T} negative semi-definite. The resulting \underline{T} is

$$\underline{T} = \begin{array}{|c|c|c|c|} \hline 0 & (1/x_1) \left(\frac{dY}{dx_1} \right) & 0 & 0 \\ \hline 0 & -a_4^2 \left(g/x_2 \right) + (a_4 f') & 0 & 0 \\ \hline 0 & 0 & -a_4 \phi - g' - \frac{1}{2} \dot{\phi} - a_4^3 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array}$$

and the corresponding \underline{S} matrix becomes

$$\underline{S} = \begin{array}{|c|c|c|c|} \hline \frac{1}{x_1^2} \left[2a_4^2 F(x_1) + Y \right] & a_4 f(x_1)/x_1 & f(x_1)/x_1 & 0 \\ \hline a_4 f(x_1)/x_1 & \frac{1}{x_2^2} \left[2a_4 G(x_2) + Y \right] & a_4^3 + g(x_2)/x_2 & a_4^2 \\ \hline f(x_1)/x_1 & a_4^3 + g(x_2)/x_2 & \phi & a_4 \\ \hline 0 & a_4^2 & a_4 & 1 \\ \hline \end{array}$$

where $f' = \frac{df(x_1)}{dx_1}$, $g' = \frac{dg(x_2)}{dx_2}$, $\phi = \phi(x_1, x_2)$, $Y = \int_0^{x_2} \left[a_4^2 \phi(x_1, x_2) - f'(x_1) \right] x_2 dx_2$, $G(x_2) = \int_0^{x_2} g(x_2) dx_2$ and $F(x_1) = \int_0^{x_1} f(x_1) dx_1$.

The corresponding Liapunov function and the time derivative are

$$V = 2a_4^2 F(x_1) + 2a_4 f(x_1)x_2 + 2f(x_1)x_3 + 2a_4 G(x_2) + \\ + 2x_3 g(x_2) + \phi(x_1, x_2)x_3^2 + 2a_4^3 x_2x_3 + 2a_4^2 x_2x_4 + \\ + 2a_4x_3x_4 + 2 \int_0^{x_2} \left[a_4^2 \phi(x_1, x_2) - f'(x_1) \right] x_2 dx_2,$$

and

$$\dot{V} = -2a_4 \left[a_4 g(x_2)/x_2 - f'(x_1) \right] x_2^2 + \\ - 2 \left[a_4 \phi(x_1, x_2) - g'(x_2) - 1/2 \dot{\phi}(x_1, x_2) - a_4^3 \right] x_3^2 + \\ + 2x_4 \int_0^{x_2} \left[a_4^2 \left\{ \frac{\partial \phi(x_1, x_2)}{\partial x_1} \right\} - f''(x_1) \right] x_2 dx_2.$$

Thus, the conditions which give asymptotic stability are

$$a_4 > 0, g(x_2)/x_2 > 0, \left[a_4 g(x_2)/x_2 - f'(x_1) \right] > 0,$$

$$\left[a_4 \phi(x_1, x_2) - g'(x_2) - 1/2 \dot{\phi}(x_1, x_2) - a_4^3 \right] > 0,$$

$$x_2 \int_0^{x_2} \left[a_4^2 \frac{\partial \phi}{\partial x_1} - f''(x_1) \right] x_2 dx_2 \leq 0$$

$$x_1 f(x_1) > 0, x_2 g(x_2) > 0, \phi(x_1, x_2) > 0,$$

$$\left[a_4 F(x_1) + x_2 f(x_1) + G(x_2) \right] > 0,$$

$$\left[f(x_1) + g(x_2) \right] x_3 \geq 0,$$

and

$$a_4^2 \phi(x_1, x_2) - f'(x_1) > 0.$$

$$(23) \quad \ddot{x} + a_3 \dot{x} + a_2 x + f(x) = 0$$

The matrix form of the equation is

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-f(x_1)}{x_1} & -a_2 & -a_3 \end{bmatrix} \underline{x}, \quad x = x_1, \quad \dot{x} = x_2, \quad \ddot{x} = x_3.$$

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{bmatrix} \frac{2a_3 F(x_1)}{x_1^2} & f(x_1)/x_1 & 0 \\ f(x_1)/x_1 & a_2 + a_3^2 & a_3 \\ 0 & a_3 & 1 \end{bmatrix} \underline{x},$$

or

$$V = 2a_3 \int_0^{x_1} f(x_1) dx_1 + 2f(x_1) x_2 + a_2 x_2^2 + (a_3 x_2 + x_3)^2,$$

and

$$\dot{V} = \underline{x}_T \underline{T} \underline{x} = 2 \underline{x}_T \begin{bmatrix} 0 & 0 & 0 \\ 0 & -[a_2 a_3 - f'(x_1)] & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x},$$

or

$$\dot{V} = -2 \left[a_2 a_3 - f'(x_1) \right] x_2^2, \quad \text{where } F(x_1) = \int_0^{x_1} f(x_1) dx_1.$$

The equilibrium solution is asymptotically stable if

$$a_2 > 0, \quad a_3 > 0, \quad [a_2 a_3 - f'(x_1)] > 0,$$

$$\left[a_3 \int_0^{x_1} f(x_1) dx_1 + f(x_1) x_2 \right] > 0,$$

and V satisfies the proper closedness properties.

(24) $\ddot{x} + a_3 \ddot{x} + h(x) \dot{x} + a_1 x = 0.$

The corresponding matrix equation is

$$\dot{\underline{x}} = \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline -a_1 & -h(x_1) & -a_3 \\ \hline \end{array}$$

$\underline{x}, x_1 = x, x_2 = \dot{x}, x_3 = \ddot{x}.$

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{array}{|c|c|c|} \hline a_1 a_3 & a_1 & 0 \\ \hline a_1 & h(x_1) + a_3^2 & a_3 \\ \hline 0 & a_3 & 1 \\ \hline \end{array} \underline{x},$$

or

$$V = a_1 (\sqrt{a_3} x_1 + x_2 / \sqrt{a_3})^2 + [h(x_1) - a_1 / a_3] x_2^2 + (a_3 x_2 + x_3)^2,$$

and

$$\dot{V} = 2 \underline{x}_T \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & -[a_3 h(x_1) - a_1 - 1/2 h'(x_1) x_2] & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \underline{x},$$

or

$$\dot{V} = -2 [a_3 h(x_1) - a_1 - 1/2 h'(x_1) x_2] x_2^2.$$

The equilibrium solution is asymptotically stable if

$$a_1 > 0, a_3 > 0, a_3 h(x) - a_1 - 1/2 x_2 h'(x_1) > 0,$$

and V satisfies the proper closedness properties.

(25) $\ddot{x} + a_3 \ddot{x} + g(\dot{x}) + a_1 x = 0$

The matrix form of the equation is

$$\dot{\underline{x}} = \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline -a_1 & -g(x_2)/x_2 & -a_3 \\ \hline \end{array}$$

$\underline{x}, x_1 = x, x_2 = \dot{x}, x_3 = \ddot{x}.$

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T$$

$a_1 a_3$	a_1	0
a_1	$\frac{2 G(x_2)}{x_2^2} + a_3^2$	a_3
0	a_3	1

 $\underline{x},$

or

$$V = a_1 \left(\sqrt{a_3} x_1 + x_2 / \sqrt{a_3} \right)^2 + (a_3 x_2 + x_3)^2 + 2 \int_0^{x_2} \left[\frac{g(x_2)}{x_2} - \frac{a_1}{a_3} \right] x_2 dx_2$$

where

$$G(x_2) = \int_0^{x_2} g(x_2) dx_2,$$

and

$$\dot{V} = 2 \underline{x}_T$$

0	0	0
0	$-[a_3 g(x_2)/x_2 - a_1]$	0
0	0	0

 $\underline{x},$

or

$$\dot{V} = -2 \left[a_3 g(x_2)/x_2 - a_1 \right] x_2^2.$$

The conditions for asymptotic stability are

$$a_1 > 0, \left[a_3 g(x_2)/x_2 - a_1 \right] > 0, a_3 > 0,$$

and the closedness of V.

(26) $\ddot{\ddot{x}} + a_3 \ddot{\ddot{x}} + g(\dot{x}) + f(x) = 0$

The matrix form of the equation is

$$\dot{\underline{x}} =$$

0	1	0
0	0	1
$-f(x_1)/x_1$	$-g(x_2)/x_2$	$-a_3$

 $\underline{x}, x_1 = x, x_2 = \dot{x}, x_3 = \ddot{x}.$

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T$$

$2a_3 F(x_1)/x_1^2$	$f(x_1)/x_1$	0
$f(x_1)/x_1$	$2G(x_2)/x_2^2 + a_3 \frac{2}{3}$	a_3
0	a_3	1

 $\underline{x},$

or

$$V = 2a_3 \int_0^{x_1} f(x_1) dx_1 + 2f(x_1) x_2 + 2 \int_0^{x_2} g(x_2) dx_2 + (a_3 x_2 + x_3)^2,$$

where

$$F(x_1) = \int_0^{x_1} f(x_1) dx_1, \quad G(x_1) = \int_0^{x_2} g(x_2) dx_2,$$

and

$$\dot{V} = \underline{x}_T \underline{T} \underline{x} = 2 \underline{x}_T$$

0	0	0
0	$-[a_3 g(x_2)/x_2 - f'(x_1)]$	0
0	0	0

 $\underline{x},$

or

$$\dot{V} = -2 \left[a_3 g(x_2)/x_2 - f'(x_1) \right] x_2^2.$$

The stability conditions are

$$a_3 > 0, \quad [a_3 g(x_2)/x_2 - f'(x_1)] > 0,$$

$$a_3 \int_0^{x_1} f(x_1) dx_1 + f(x_1) x_2 + \int_0^{x_2} g(x_2) dx_2 > 0.$$

$$(27) \quad \ddot{x} + a_3 \dot{x} + g(x, \dot{x}) + f(x) = 0.$$

The matrix form of the equation is

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{f(x_1)}{x_1} & -\frac{g(x_1, x_2)}{x_2} & -a_3 \end{bmatrix} \underline{x}$$

$$\underline{x}, x_1 = x, x_2 = \dot{x}, x_3 = \ddot{x}.$$

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{bmatrix} 2a_3 F(x_1)/x_1^2 & f(x_1)/x_1 & 0 \\ f(x_1)/x_1 & 2G(x_1, x_2)/x_2^2 + a_3^2 & a_3 \\ 0 & a_3 & 1 \end{bmatrix} \underline{x},$$

or

$$V = 2a_3 \int_0^{x_1} f(x_1) dx_1 + 2 \int_0^{x_2} g(x_1, x_2) dx_2 + 2 f(x_1) x_2 + (a_3 x_2 + x_3)^2,$$

where

$$F(x_1) = \int_0^{x_1} f(x_1) dx_1, G(x_1, x_2) = \int_0^{x_2} g(x_1, x_2) dx_2$$

and

$$\dot{V} = 2 \underline{x}_T \begin{bmatrix} 0 & 0 & 0 \\ 0 & - \left[a_3 \frac{g(x_1, x_2)}{x_2} - f'(x_1) + \frac{1}{x_2} \int_0^{x_2} \frac{\partial g(x_1, x_2)}{\partial x_1} dx_2 \right] & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x},$$

or

$$\dot{V} = -2 \left[a_3 \frac{g(x_1, x_2)}{x_2} - f'(x_1) + \frac{1}{x_2} \int_0^{x_2} \frac{\partial g(x_1, x_2)}{\partial x_1} dx_2 \right] x_2^2.$$

The stability conditions are

$$a_3 > 0, \left[a_3 F(x_1) + f(x_1)x_2 + G(x_1, x_2) \right] > 0,$$

$$\left[a_3 \frac{g(x_1, x_2)}{x_2} - f'(x_1) + \frac{1}{x_2} \int_0^{x_2} \frac{\partial g(x_1, x_2)}{\partial x_1} dx_2 \right] > 0.$$

$$(28) \quad \ddot{x} + \gamma(x, \dot{x}) \dot{x} + a_2 \dot{x} + f(x) = 0.$$

The matrix form is given by

$$\dot{\underline{x}} = \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline -f(x_1)/x_1 & -a_2 & -\gamma(x_1, x_2) \\ \hline \end{array} \quad \underline{x}, \quad x_1 = x, x_2 = \dot{x}, x_3 = \ddot{x}.$$

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{array}{|c|c|c|} \hline \frac{2a_3 F + a_3 Y}{x_1^2} & f(x_1)/x_1 & 0 \\ \hline f(x_1)/x_1 & a_2 + a_3 \frac{Y}{x_2^2} & a_3 \\ \hline 0 & a_3 & 1 \\ \hline \end{array} \underline{x},$$

or

$$V = 2a_3 \int_0^{x_1} f(x_1) dx_1 + 2 f(x_1) x_2 + a_2 x_2^2 +$$

$$+ 2a_3 \int_0^{x_2} [\gamma(x_1, x_2) - a_3] x_2 dx_2 + (a_3 x_2 + x_3)^2,$$

where

$$F = \int_0^{x_1} f(x_1) dx_1, \quad Y = \int_0^{x_2} \gamma(x_1, x_2) x_2 dx_2$$

and

$$\dot{V} = -2 \left[a_3 a_2 - f'(x_1) \right] x_2^2 - 2 \left[\gamma(x_1, x_2) - a_3 \right] x_3^2 + \\ + 2 a_3 x_2 \int_0^{x_2} \left[\frac{\partial \gamma(x_1, x_2)}{\partial x_1} \right] x_2 dx_2.$$

The stability conditions are

$$a_3 > 0, a_2 > 0, a_3 \int_0^{x_1} f(x_1) dx_1 + f(x_1) x_2 > 0, \\ \left[a_2 a_3 - f'(x_1) \right] > 0, \left[\gamma(x_1, x_2) - a_3 \right] > 0,$$

and

$$x_2 \int_0^{x_2} \left[\frac{\partial \gamma(x_1, x_2)}{\partial x_1} \right] x_2 dx_2 < 0.$$

(29) $\ddot{x} + \gamma(x, \dot{x}) \ddot{x} + g(\dot{x}) + a_1 x = 0.$

The matrix form is given by

$$\dot{\underline{x}} = \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline -a_1 & -g(x_2)/x_2 & -\gamma \\ \hline \end{array}$$

\underline{x} , where $\gamma = \gamma(x_1, x_2)$.

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{array}{|c|c|c|} \hline a_1 a_3 + a_3 Y / X_1^2 & a_1 & 0 \\ \hline a_1 & \frac{2G(x_2) + a_3 Y}{x_2^2} & a_3 \\ \hline 0 & a_3 & 1 \\ \hline \end{array} \underline{x},$$

or

$$V = a_1 (\sqrt{a_3} x_1 + x_2 / \sqrt{a_3})^2 + (a_3 x_2 + x_3)^2 + 2 \int_0^{x_2} \left[\frac{g(x_2)}{x_2} - \frac{a_1}{a_3} \right] x_2 dx_2 + \\ + 2 a_3 \int_0^{x_2} [\gamma(x_1, x_2) - a_3] x_2 dx_2,$$

where

$$G(x_2) = \int_0^{x_2} g(x_2) dx_2, \quad Y = \int_0^{x_2} \gamma(x_1, x_2) x_2 dx_2,$$

and

$$\dot{V} = -2 \left[a_3 g(x_2)/x_2 - a_1 \right] x_2^2 - 2 [\gamma(x_1, x_2) - a_3] x_3^2 + \\ + a_3 a_2 \int_0^{x_2} \left[\frac{\partial \gamma(x_1, x_2)}{\partial x_1} \right] x_2 dx_2.$$

The stability conditions are

$$a_3 > 0, \quad \left[g(x_2)/x_2 - a_1/a_3 \right] > 0, \quad [\gamma(x_1, x_2) - a_3] > 0,$$

$$x_2 \int_0^{x_2} \left[\frac{\partial \gamma(x_1, x_2)}{\partial x_1} \right] x_2 dx_2 < 0, \quad a_1 > 0.$$

$$(30) \quad \ddot{\underline{x}} + \gamma(x, \dot{x}) \dot{\underline{x}} + g(\dot{x}) + f(x) = 0.$$

The matrix form is given by

$$\dot{\underline{x}} = \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline -f(x_1)/x_1 & -g(x_2)/x_2 & -\gamma \\ \hline \end{array}$$

\underline{x} , where $\gamma = \gamma(x_1, x_2)$.

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T$$

$\frac{2a_3 F(x_1) + a_3 Y}{x_1^2}$	$f(x_1)/x_1$	0
$f(x_1)/x_1$	$\frac{2G(x_2) + a_3 Y}{x_2^2}$	a_3
0	a_3	1

 $\underline{x},$

or

$$V = (a_3 x_2 + x_3)^2 + 1/a_2 \left[a_2 x_2 + f(x_1) \right]^2 + 2 \int_0^{x_2} \left[g(x_2)/x_2 - a_2 \right] x_2 dx_2 +$$

$$+ 2a_3 \int_0^{x_2} \left[\gamma(x_1, x_2) - a_3 \right] x_2 dx_2 + \frac{1}{a_2} \left[2a_2 a_3 F(x_1) - f^2(x_1) \right],$$

where

$$F(x_1) = \int_0^{x_1} f(x_1) dx_1, \quad G(x_2) = \int_0^{x_2} g(x_2) dx_2,$$

$$Y(x_1, x_2) = \int_0^{x_2} \gamma(x_1, x_2) x_2 dx_2,$$

and

$$\dot{V} = -2 \left[a_3 g(x_2)/x_2 - f'(x_1) \right] x_2^2 - 2 \left[\gamma(x_1, x_2) - a_3 \right] x_3^2 +$$

$$+ 2a_3 x_2 \int_0^{x_2} \left[\frac{\partial \gamma(x_1, x_2)}{\partial x_1} \right] x_2 dx_2.$$

The stability conditions are

$$a_2 > 0, \quad a_3 > 0, \quad \left[a_3 g(x_2)/x_2 - f'(x_1) \right] > 0,$$

$$\gamma(x_1, x_2) - a_3 > 0, \quad x_2 \int_0^{x_2} \left[\frac{\partial \gamma(x_1, x_2)}{\partial x_1} \right] x_2 dx_2 < 0,$$

and

$$\left[\frac{g(x_2)}{x_2} - a_2 \right] > 0, \quad \left[2a_2a_3F(x_1) - f^2(x_1) \right] > 0.$$

$$(31) \quad \ddot{\ddot{x}} + a_4 \ddot{\ddot{x}} + a_3 \ddot{\ddot{x}} + a_2 \dot{\ddot{x}} + f(x) = 0.$$

The matrix form of the equation is

$$\dot{\underline{x}} = \begin{array}{c|ccc} & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{f(x_1)}{x_1} & -a_2 & -a_3 & -a_4 & \end{array}$$

$$\underline{x}, \quad x_1 = x, x_2 = \dot{x}, x_3 = \ddot{x}, x_4 = \ddot{\ddot{x}}.$$

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{array}{c|ccc} \frac{2 \Delta F(x_1)}{x_1^2} & a_4 f(x_1)/x_1 & f(x_1)/x_1 & 0 \\ \hline a_4 f(x_1)/x_1 & a_2 a_4 + a_3 \Delta - f'(x_1) & a_3 a_4 & \Delta \\ f(x_1)/x_1 & a_3 a_4 & a_4^2 + a_2/a_4 & a_4 \\ 0 & \Delta & a_4 & 1 \end{array} \underline{x},$$

where

$$\Delta = (a_3 - a_2/a_4), \quad F(x_1) = \int_0^{x_1} f(x_1) dx_1, \quad f'(x_1) = \frac{df(x_1)}{dx_1},$$

or

$$V = (x_4 + a_4 x_3 + \Delta x_2)^2 + \frac{a_2}{a_4} (x_3 + a_4 x_2 + a_4/a_2 f(x_1))^2 + \\ + 1/a_4 (a_2 \Delta - a_4 f'(x_1) x_2^2 + 1/a_2 (2a_2 \Delta F(x_1) - a_4 f^2(x_1))),$$

and

$$\dot{V} = -2 \left[a_2 \Delta - a_4 f'(x_1) + 1/2 f''(x_1) x_2 \right] x_2^2.$$

The stability conditions are

$$a_2 > 0, a_4 > 0, \Delta > 0, \left[a_2 \Delta - a_4 f'(x_1) \right] > 0,$$

$$\left[2a_2 \Delta f(x_1) - a_4 f^2(x_1) \right] > 0,$$

and

$$\left[a_2 \Delta - a_4 f'(x_1) + 1/2 f''(x_1) x_2 \right] > 0.$$

$$(32) \quad \ddot{x} + a_4 \ddot{x} + a_3 \ddot{x} + g(\dot{x}) + a_1 x = 0.$$

The matrix form of the equation is

$$\dot{\underline{x}} = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline -a_1 & -g(x_2)/x_2 & -a_3 & -a_4 \\ \hline \end{array} \underline{x}, \quad x_1 = x, x_2 = \dot{x}, x_3 = \ddot{x}, x_4 = \dddot{x}.$$

The corresponding Liapunov function is

$$V = \underline{x}_T \begin{array}{|c|c|c|c|} \hline a_1 & a_1 a_4 & a_1 & 0 \\ \hline a_1 a_4 & a_3 \Delta - a_1 + \frac{2a_4 G}{x_2} & a_4 \Delta + g(x_2)/x_2 & \Delta \\ \hline a_1 & a_4 \Delta + g(x_2)/x_2 & a_4^2 + a_2/a_4 & a_4 \\ \hline 0 & \Delta & a_4 & 1 \\ \hline \end{array} \underline{x},$$

or

$$V = (\Delta x_2 + a_4 x_3 + x_4)^2 + a_2/a_4 (x_3 + a_4 x_2 + \frac{a_1 a_4 x_1}{a_2})^2 +$$

$$+ 2 \left[g(x_2) x_3 - a_2 x_2 x_3 \right] + 1/a_4 \left[a_2 \Delta - a_1 a_4 \right] \left[x_2^2 + \frac{a_1 a_4}{a_2} x_1^2 \right] +$$

$$+ 2a_4 \int_0^{x_2} \left[g(x_2)/x_2 - a_2 \right] x_2 dx_2,$$

where

$$\Delta = a_3 - a_2/a_4, \quad G(x_2) = \int_0^{x_2} g(x_2) dx_2$$

and

$$\dot{V} = -2 \left[\frac{g(x_2)}{x_2} \Delta - a_4 a_1 \right] x_2^2 - \left[a_2 - g'(x_2) \right] x_3^2.$$

The stability conditions are

$$a_1 > 0, a_2 > 0, a_4 > 0, \Delta > 0,$$

$$\left[a_2 \Delta - a_1 a_4 \right] > 0, \quad g(x_2) x_3 - a_2 x_2 x_3 > 0,$$

$$\left[\frac{g(x_2)}{x_2} \Delta - a_1 a_4 \right] > 0, \quad \left[\frac{g(x_2)}{x_2} - a_2 \right] > 0,$$

and

$$a_2 - g'(x_2) > 0.$$

$$(33) \quad \ddot{\ddot{x}} + a_4 \ddot{\ddot{x}} + \gamma(x, \dot{x}) \ddot{x} + a_2 \dot{x} + a_1 x = 0.$$

The matrix form of the equation is

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_1 & -a_2 & -\gamma & -a_4 \end{bmatrix} \underline{x}, \text{ where } \gamma = \gamma(x_1, x_2).$$

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{bmatrix} \frac{a_1^2 a_4}{a_2} + \frac{a_1 a_4 \gamma}{a_2 x_1^2} & a_4 a_1 & a_1 & 0 \\ a_4 a_1 & a_4 a_2 - a_1 + \frac{a_1 a_4 \gamma}{a_2 x_1^2} & \frac{a_4 a_1}{a_2} + a_2 & \frac{a_4 a_1}{a_2} \\ a_1 & \frac{a_4 a_1}{a_2} + a_2 & a_4^2 - \frac{a_1 a_4}{a_2} + \gamma & a_4 \\ 0 & \frac{a_4 a_1}{a_2} & a_4 & 1 \end{bmatrix} \underline{x}$$

or

$$V = \left[x_4 + a_4 x_3 + \frac{a_1 a_4 x_2}{a_2} \right]^2 x_2^2 + \frac{a_2}{a_4} \left[x_3 + a_4 x_2 + \frac{a_1 a_4 x_1}{a_2} \right]^2 +$$

$$+ \frac{1}{a_2 a_4} \left[a_2 a_4 \psi(x_1, x_2) - a_2^2 - a_1 a_4 \right] x_3^2 +$$

$$+ \frac{2a_1}{a_2^2} \left\{ \int_0^{x_2} \left[a_2 a_4 \psi(x_1, x_2) - a_2^2 - a_1 a_4 \right] x_2 dx_2 \right\},$$

where

$$Y = \int_0^{x_2} \psi(x_1, x_2) x_2 dx_2,$$

and

$$\dot{V} = - \frac{2}{a_2} \left[a_2 a_4 \psi(x_1, x_2) - a_2^2 - a_1 a_4 - \frac{a_2}{2} \dot{\psi} \right] x_3^2 +$$

$$+ \frac{a_1 a_4 x_2}{a_2} \int_0^{x_2} \frac{\partial \psi(x_1, x_2)}{\partial x_1} x_2 dx_2,$$

where $\dot{\psi} = \frac{\partial \psi}{\partial x_1} x_2 + \frac{\partial \psi}{\partial x_2} x_3$. The conditions for stability are

$$a_1 > 0, a_2 > 0, a_4 > 0,$$

$$a_2 a_4 \psi - a_2^2 - a_1 a_4 - a_2^2 / 2 \dot{\psi} > 0$$

and

$$a_2 a_4 \psi - a_2^2 - a_1 a_4 > 0, \quad \frac{\partial \psi}{\partial x_1} < 0.$$

$$(34) \quad \ddot{\ddot{x}} + a_4 \ddot{\ddot{x}} + a_3 \ddot{x} + g(x) + f(x) = 0.$$

The matrix form of the equation is

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{f(x_1)}{x_1} & -\frac{g(x_2)}{x_2} & -a_3 & -a_4 \end{bmatrix} \underline{x}.$$

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T$$

$\frac{2 \Delta F(x_1)}{x_1}$	$a_4 f(x_1)/x_1$	$f(x_1)/x_1$	0
$a_4 f(x_1)/x_1$	$a_3 \Delta - f'(x_1) + \frac{2a_4}{x_2} G(x_2)$	$a_4 \Delta + g(x_2)/x_2$	Δ
$f(x_1)/x_1$	$a_4 \Delta + g(x_2)/x_2$	$\frac{2}{a_4} + \frac{a_2}{a_4}$	a_4
0	Δ	a_4	1

or

$$V = (x_4 + a_4 x_3 + \Delta x_2)^2 + \frac{a_2}{a_4} (x_3 + a_4 x_2 + a_4/a_2 f(x_1))^2 +$$

$$+ \frac{1}{a_4} (a_2 \Delta - a_4 f'(x_1)) x_2^2 + \frac{1}{a_2} (2a_2 \Delta F(x_1) - a_4 f'^2(x_1))$$

$$+ 2 a_4 \int_0^{x_2} \left[\frac{g(x_2)}{x_2} - a_2 \right] x_2 dx_2 + 2 \left[g(x_2) x_3 - a_2 x_2 x_3 \right],$$

where

$$\Delta = a_3 - a_2/a_4, \quad f'(x_1) = df/dx_1,$$

$$F(x_1) = \int_0^{x_1} f(x_1) dx_1,$$

and

$$G(x_2) = \int_0^{x_2} g(x_2) dx_2,$$

and

$$\dot{V} = -2 \left[\frac{g(x_2)}{x_2} \Delta - a_4 f'(x_1) + 1/2 f''(x_1) x_2 \right] x_2^2 - 2 \left[a_2 - g'(x_2) \right] x_3^2,$$

where

$$f'' = \frac{d^2 f}{dx_1^2} \quad \text{and} \quad g' = \frac{dg}{dx_2}.$$

The conditions of stability are

$$a_2 > 0, a_4 > 0, \Delta > 0, [a_2 \Delta - a_4 f'] > 0,$$

$$[2a_2 \Delta F - a_4 f^2] > 0, g(x_2) x_3 - a_2 x_2 x_3 > 0,$$

$$\left[\frac{g(x_2)}{x_2} - a_2 \right] > 0, a_2 - g'(x_2) > 0,$$

and

$$\left[\frac{g(x_2)}{x_2} \Delta - a_4 f'(x_1) + \frac{1}{2} f''(x_1) x_2 \right] > 0.$$

$$(35) \quad \ddot{x} + a_4 \ddot{x} + a_3 \ddot{x} + g(x, \dot{x}) + f(x) = 0.$$

The matrix equation is

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{f(x_1)}{x_1} & -\frac{g(x_1, x_2)}{x_2} & -a_3 & -a_4 \end{bmatrix} \underline{x}.$$

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{bmatrix} \frac{2 \Delta F(x_1)}{2 x_1} & a_4 \frac{f(x_1)}{x_1} & \frac{f(x_1)}{x_1} & 0 \\ a_4 \frac{f(x_1)}{x_1} & a_3 \Delta - f'(x_1) + \frac{2a_4 G}{x_2^2} & a_4 \Delta + \frac{g(x_1, x_2)}{x_2} & \Delta \\ \frac{f(x_1)}{x_1} & a_4 \Delta + \frac{g(x_1, x_2)}{x_2} & a_4^2 + \frac{a_2}{a_4} & a_4 \\ 0 & \Delta & a_4 & 1 \end{bmatrix} \underline{x},$$

or

$$V = (x_4 + a_4 x_3 + \Delta x_2)^2 + \frac{a_2}{a_4} (x_3 + a_4 x_2 + \frac{a_4}{a_2} f(x_1))^2 + \\ + 1/a_4 (a_2 \Delta - a_4 f'(x_1)) x_2^2 + 1/a_2 (2a_2 \Delta F(x_1) - a_4 f^2(x_1)) + \\ + 2 a_4 \int_0^{x_2} \left(\frac{g(x_1, x_2)}{x_2} - a_2 \right) x_2 dx_2 + 2 (g(x_1, x_2) - a_2 x_2 x_3),$$

where

$$F = \int_0^{x_1} f(x_1) dx_1, \quad G = \int_0^{x_2} g(x_1, x_2) dx_2,$$

$$f' = df(x_1)/dx_1, \quad \Delta = a_3 - a_2/a_4,$$

and

$$\begin{aligned} \dot{V} = & -2 \left[\frac{g(x_1, x_2)}{x_2} \Delta - a_4 f'(x_1) + 1/2 f''(x_1) x_2 + \right. \\ & - 1/x_2 \frac{\partial g(x_1, x_2)}{\partial x_1} - \frac{a_4}{x_2} \int_0^{x_2} \left[\frac{\partial g(x_1, x_2)}{\partial x_1} \right] dx_2 \Big] x_2^2 + \\ & - \left[a_2 - \frac{\partial g(x_1, x_2)}{\partial x_2} \right] x_2^3. \end{aligned}$$

The conditions for stability are

$$a_2 > 0, a_4 > 0, \Delta > 0, a_2 \Delta - a_4 f'(x_1) > 0,$$

$$2a_2 \Delta F(x_1) - a_4 f^2(x_1) > 0, \quad g(x_1, x_2) > a_2 x_2 x_3,$$

$$\left[\frac{g(x_1, x_2)}{x_2} - a_2 \right] > 0, \quad \left[a_2 - \frac{\partial g(x_1, x_2)}{\partial x_2} \right] > 0,$$

and

$$\begin{aligned} & \left[\frac{g(x_1, x_2)}{x_2} \Delta - a_4 f'(x_1) + 1/2 f''(x_1) x_2 - 1/x_2 \frac{\partial g(x_1, x_2)}{\partial x_1} + \right. \\ & \left. - \frac{a_4}{x_2} \int_0^{x_2} \frac{\partial g(x_1, x_2)}{\partial x_1} dx_2 \right] > 0. \end{aligned}$$

$$(36) \quad \ddot{x} + a_4 \ddot{x} + \gamma(x, \dot{x}) \ddot{x} + a_2 \dot{x} + f(x) = 0.$$

The matrix form of the equation is

$$\underline{\ddot{x}} = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline -f(x_1)/x_1 & -a_2 & -\gamma & -a_4 \\ \hline \end{array}$$

$\underline{\ddot{x}}$, where $\gamma = \gamma(x_1, x_2)$.

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T$$

$\frac{1}{x_2^2} \left[\frac{a_1 a_4 F(x_1)}{a_2} + Y \right]$	$a_4 \frac{f(x_1)}{x_1}$	$\frac{f(x_1)}{x_1}$	0
$a_4 \frac{f(x_1)}{x_1}$	$a_2 a_4 + \frac{Y}{x_2^2}$	$\frac{a_1 a_4}{a_2} + a_2$	$\frac{a_1 a_4}{a_2}$
$\frac{f(x_1)}{x_1}$	$\frac{a_1 a_4}{a_2} + a_2$	$a_4^2 - \frac{a_1 a_4}{a_2} + \gamma$	a_4
0	$\frac{a_1 a_4}{a_2}$	a_4	1

or

$$V = (x_4 + a_4 x_3 + \frac{a_1 a_4}{a_2} x_2^2)^2 + \frac{a_2}{a_4} (x_3 + a_4 x_2 + \frac{a_4}{a_2} f(x_1))^2 +$$

$$+ \frac{1}{a_2 a_4} (a_2 a_4 \gamma(x_1, x_2) - a_2^2 - a_1 a_4) x_3^2 +$$

$$+ \frac{1}{a_2 a_4} (a_1 a_4 F(x_1) - a_4^2 f(x_1)) +$$

$$+ \frac{2a_1}{a_2^2} \left\{ \int_0^{x_2} \left[a_2 a_4 \gamma(x_1, x_2) + (-a_2^2/a_1) f'(x_1) - a_1 a_4^2 \right] x_2 dx_2 \right\},$$

where

$$F = \int_0^{x_1} f(x_1) dx_1, \quad Y = \int_0^{x_2} \left[\frac{a_1 a_4}{a_2} \gamma - f'(x_1) \right] x_2 dx_2,$$

and

$$\dot{V} = -2a_4 \left[a_1 - f'(x_1) \right] x_2^2 - \frac{2}{a_2} \left[a_4 a_2 \gamma(x_1, x_2) + \right.$$

$$\left. - a_2^2 - a_1 a_4^2 - \frac{1}{2} a_2 \dot{\gamma}(x_1, x_2) \right] x_3^2 +$$

$$+ 2x_2 \int_0^{x_2} \left[\frac{a_1 a_4}{a_2} \frac{\partial \gamma(x_1, x_2)}{\partial x_1} - f''(x_1) \right] x_2 dx_2,$$

where

$$\frac{\partial Y}{\partial x_1} = \int_0^{x_2} \left[\frac{a_1 a_4}{a_2} \frac{\partial \psi}{\partial x_1} - f'' \right] x_2 dx_2,$$

and

$$\dot{\psi} = \left(\frac{\partial \psi}{\partial x_1} \right) x_2 + \left(\frac{\partial \psi}{\partial x_2} \right) x_3.$$

The conditions of stability are

$$a_1 > 0, a_2 > 0, a_4 > 0, a_2 a_4 \psi - a_2^2 - a_1 a_4^2 - \frac{a}{2} \dot{\psi}(x_1, x_2) > 0,$$

$$a_1 - f'(x_1) > 0, a_2 a_4 \psi - a^2 \frac{2}{a_1} f' - a_4^2 a_1 > 0,$$

$$\left[\frac{a_1 a_4}{a_2} \frac{\partial \psi}{\partial x_1} - f'' \right] < 0, a_2 a_4 \psi - a^2 \frac{2}{a_1} - a_1 a_4^2 > 0,$$

$$a_1 a_4^2 F(x_1) - a_4^2 f^2(x_1) > 0.$$

$$(37) \quad \ddot{\ddot{x}} + a_4 \ddot{\ddot{x}} + \psi(x, \dot{x}) \ddot{x} + g(\dot{x}) + a_1 x = 0.$$

The matrix form of the equation is

$$\dot{\underline{x}} = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline -a_1 & \frac{-g(x_2)}{x_2} & -\psi & -a_4 \\ \hline \end{array}$$

\underline{x} , where $\psi = \psi(x_1, x_2)$.

The corresponding Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{array}{|c|c|c|c|} \hline \frac{a_1^2 a_4}{a_2} + \frac{Y}{x_1^2} & a_1 a_4 & a_1 & 0 \\ \hline a_1 a_4 & \frac{1}{x_2^2} [2a_4 G(x_2) + Y] & \frac{a_1 a_4}{a_2} + \frac{g(x_2)}{x_2} & \frac{a_1 a_4}{a_2} \\ \hline a_1 & \frac{a_1 a_4^2}{a_2} + \frac{g(x_2)}{x_2} & a_4^2 - \frac{a_1 a_4}{a_2} + \psi & a_4 \\ \hline 0 & \frac{a_1 a_4}{a_2} & a_4 & 1 \\ \hline \end{array} \underline{x},$$

or

$$\begin{aligned}
 V = & (x_4 + a_4 x_3 + \frac{a_1 a_4}{a_2} x_2)^2 + \frac{a_2}{a_4} (x_3 + a_4 x_2 + \frac{a_1 a_4}{a_2} x_1)^2 + \\
 & + \frac{1}{a_2 a_4} \left[a_2 a_4 \psi(x_1, x_2) - a_2^2 - a_1 a_4^2 \right] x_3^2 + \\
 & + 2 a_4 \int_0^{x_2} \left[\frac{g(x_2)}{x_2} - a_2 \right] x_2 dx_2 + 2 \left[g(x_2) x_3 - a_2 x_2 x_3 \right] + \\
 & + \frac{2 a_1}{a_2} \int_0^{x_2} \left[a_2 a_4 \psi(x_1, x_2) - a_2^2 - a_1 a_4^2 \right] x_2 dx_2,
 \end{aligned}$$

where

$$G(x_2) = \int_0^{x_2} g(x_2) dx_2, \quad \psi = \psi(x_1, x_2),$$

$$Y = \int_0^{x_2} \left[\frac{a_1 a_4}{a_2} \psi - a_1 \right] x_2 dx_2,$$

and

$$\begin{aligned}
 \dot{V} = & - 2 \frac{a_1 a_4}{a_2} \left[\frac{g(x_2)}{x_2} - a_2 \right] x_2^2 - \frac{2 a_1 a_4}{a_2} x_2 \int_0^{x_2} \frac{\partial \psi(x_1, x_2)}{\partial x_1} x_2 dx_2 + \\
 & - \frac{2}{a_2} \left[a_2 a_4 \psi - a_2 g' - a_1 a_4^2 - \frac{a_2}{2} \dot{\psi} \right] x_3^2,
 \end{aligned}$$

where

$$\dot{\psi} = \frac{\partial \psi}{\partial x_1} x_2 + \frac{\partial \psi}{\partial x_2} x_3.$$

The conditions for stability are

$$a_1 > 0, \quad a_2 > 0, \quad a_4 > 0, \quad \frac{g(x_2)}{x_2} - a_2 > 0,$$

$$a_2 a_4 \psi(x_1, x_2) - a_2^2 - a_1 a_4^2 > 0,$$

$$g(x_2) x_3 - a_2 x_2 x_3 > 0,$$

$$x_2 \int_0^{x_2} \frac{\partial \psi}{\partial x_1} x_2 dx_2 < 0$$

and

$$a_2 a_4 \psi - a_2 g' - a_1 a_4^2 - \frac{a_2}{2} \dot{\psi} > 0.$$

This completes those examples found in Mekel's thesis, [11].

The next example comes from Puri, [12], and is a time-varying linear system.

$$(38) \quad \ddot{x} + a_3(t) \dot{x} + a_2(t) \dot{x} + a_1(t) x = 0.$$

The matrix form of the equation is

$$\underline{\dot{x}} = \underline{A}(t) \underline{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \underline{x}; \quad x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = \ddot{x}.$$

The form of the Liapunov function is

$$V = \underline{x}_T \underline{S} \underline{x} = \underline{x}_T \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & 1 \end{bmatrix} \underline{x},$$

where the time derivative is given by

$$\begin{aligned} \dot{V} &= \dot{\underline{x}}_T \underline{S} \underline{x} + \underline{x}_T \dot{\underline{S}} \underline{x} + \underline{x}_T \underline{S} \dot{\underline{x}} = \underline{x}_T \left[\underline{\dot{A}}_T \underline{S} + \dot{\underline{S}} + \underline{S} \underline{A} \right] \underline{x} = \\ &= \underline{x}_T \left[2 \underline{S} \underline{A} + \dot{\underline{S}} \right] \underline{x} = \underline{x}_T \underline{T} \underline{x} = \end{aligned}$$

$$= \underline{x}_T \begin{bmatrix} \dot{S}_{11} - 2a_1 S_{13} & 2S_{11} - 2a_2 S_{13} - 2a_1 S_{23} + 2\dot{S}_{12} & 2S_{12} - 2a_1 S_{13} - 2a_1 + 2\dot{S}_{13} \\ 0 & \dot{S}_{22} + 2S_{12} - 2a_2 S_{23} & 2S_{22} - 2a_1 S_{23} + 2S_{13} + -2a_2 + 2\dot{S}_{23} \\ 0 & 0 & 2S_{23} - 2a_1 \end{bmatrix} \underline{x}.$$

Let the off-diagonal elements in the matrix \underline{T} equal zero. To simplify the T_{13} element let S_{13} be identically zero. Then, $S_{12} = a_1$. To simplify the T_{12} element let $S_{23} = \alpha_3 = \text{constant}$. Then, $S_{11} = \alpha_3 a_1 - \dot{a}_1$. From the T_{23} element, $S_{22} = \alpha_3 a_3 + a_2$. The resulting Liapunov function is

$$V = \underline{x}^T \begin{bmatrix} \alpha_3 a_1 - \dot{a}_1 & a_1 & 0 \\ a_1 & \alpha_3 a_3 + a_2 & \alpha_3 \\ 0 & \alpha_3 & 1 \end{bmatrix} \underline{x},$$

or

$$V = (\alpha_3 a_1 - \dot{a}_1) x_1^2 + 2a_1 x_1 x_2 + (a_2 + \alpha_3 a_3) x_2^2 + 2\alpha_3 x_2 x_3 + x_3^2,$$

and

$$\dot{V} = \underline{x}^T \underline{T} \underline{x} = \underline{x}^T \begin{bmatrix} \alpha_3 \dot{a}_1 - \ddot{a}_1 & 0 & 0 \\ 0 & 2a_1 + \dot{a}_2 + \alpha_3 \dot{a}_3 - 2\alpha_3 a_2 & 0 \\ 0 & 0 & 2\alpha_3 - 2a_3 \end{bmatrix} \underline{x},$$

$$= (\alpha_3 \dot{a}_1 - \ddot{a}_1) x_1^2 + (2a_1 + \dot{a}_2 + \alpha_3 \dot{a}_3 - 2\alpha_3 a_2) x_2^2 + (2\alpha_3 - 2a_3) x_3^2.$$

The conditions for asymptotic stability are

$$\alpha_3 \dot{a}_1 - \ddot{a}_1 < 0, \quad t \geq 0,$$

$$2a_1 + \dot{a}_2 + \alpha_3 \dot{a}_3 - 2\alpha_3 a_2 < 0, \quad t \geq 0,$$

$$\alpha_3 - a_3 < 0, \quad t \geq 0,$$

$$\alpha_3 a_1 - \dot{a}_1 > 0, \quad t \geq 0,$$

$$(\alpha_3 a_1 - \dot{a}_1)(\alpha_3 a_3 + a_2) - a_1^2 > \alpha_3^2 (\alpha_3 a_1 - \dot{a}_1).$$

The following five nonlinear, autonomous systems were analyzed by the "common Liapunov function" technique of Goldwyn and Norendra, [15] .

$$(39) \quad \ddot{x} + f(x, \dot{x})\dot{x} + g(x)x = 0.$$

The matrix form of the equation is

$$\dot{\underline{x}} = \underline{A} \underline{x} = \begin{bmatrix} 0 & 1 \\ -g & -f \end{bmatrix} \underline{x} ; \quad x_1 = x, \quad x_2 = \dot{x}, \quad f = f(x_1, x_2),$$

and $g = g(x_1)$. The Liapunov function for the corresponding linear system, where $f, g \geq 0$, is

$$V = \underline{x}_T \underline{P} \underline{x} = \underline{x}_T \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \underline{x},$$

and

$$\dot{V} = \underline{x}_T \left[\underline{A}_T \underline{P} + \underline{P} \underline{A} \right] \underline{x} = - \underline{x}_T \underline{Q} \underline{x},$$

where

$$\underline{Q} = \begin{bmatrix} 2 g P_{12} & g P_{22} - P_{11} + f P_{12} \\ g P_{22} - P_{11} + f P_{12} & 2 f P_{22} - 2 P_{12} \end{bmatrix}.$$

For constant f and g , the Routh-Hurwitz conditions demand $f, g \geq 0$ for stability. Hence $f = 0$ is in the allowable range. If f is set equal to zero, then

$$\underline{Q} = \begin{bmatrix} 2 g P_{12} & g P_{22} - P_{11} \\ g P_{22} - P_{11} & - 2 P_{12} \end{bmatrix}.$$

Since for $g > 0$, \underline{Q} can never be positive definite if \underline{P} is positive definite: thus we must take $\underline{Q} = \underline{0}$. This results in the following relationships

$$P_{12} = 0, \quad P_{11} = P_{22} g,$$

where P_{22} is assumed to be 1. Therefore, we have a CLF for the linear system when f and g are nonnegative constants. The corresponding Liapunov function for the nonlinear system is

$$V = 2 \int_0^{x_1} x_1 g(x_1) dx_1 + x_2^2,$$

and

$$\dot{V} = -2 f(x_1, x_2) x_2^2,$$

where $2 \int_0^{x_1} x_1 g(x_1) dx_1 = g x_1^2$, if g is a constant.

The conditions for asymptotic stability in the large are:

$$f(x_1, x_2) > 0, \quad g(x_1) \neq 0 \text{ for } x_1 \neq 0,$$

$$\int_0^{x_1} x_1 g(x_1) dx_1 \longrightarrow \infty \text{ as } |x_1| \longrightarrow \infty.$$

(40) $\ddot{x} + f(x) \dot{x} + g(x) x = 0.$

This example is a special case of example (39) but the state variables are chosen such that the "average dissipation", $\hat{f}(x) = \frac{\int_0^{x_1} f(x_1) dx_1}{x_1}$,

is more significant than the "instantaneous dissipation", $f(x)$. This method is called Lienárd's transformation and the resulting matrix equation is

$$\dot{\underline{x}} = \begin{array}{|c|c|} \hline -\hat{f}(x_1) & 1 \\ \hline -g(x_1) & 0 \\ \hline \end{array} \quad \underline{x}, x_1 = x.$$

For this case, the \underline{Q} matrix for a system where \hat{f} and g are assumed constant is

$$\underline{Q} = \begin{array}{|c|c|} \hline a(\hat{f} P_{11} + g P_{12}) & \hat{f} P_{12} + g P_{22} - P_{11} \\ \hline \hat{f} P_{12} + g P_{22} - P_{11} & -2 P_{12} \\ \hline \end{array}.$$

Now if we select Q to be

$$Q = \begin{bmatrix} 2(\hat{f} P_{11} + g P_{12}) & 0 \\ 0 & 0 \end{bmatrix},$$

we have a positive semi-definite matrix for

$$\hat{f} P_{11} + g P_{12} > 0.$$

From the zero terms, we find

$$P_{12} = 0, \quad P_{11} = g P_{22}, \quad \text{and} \quad \hat{f} g P_{22} > 0.$$

Let $P_{22} = \text{constant} = 1$. Thus, we have a CLF for the linear system. For the nonlinear system we have:

$$V = 2 \int_0^{x_1} x_1 g(x_1) dx_1 + x_2^2,$$

$$\dot{V} = -2 g \hat{f} x_1^2.$$

From V and \dot{V} we obtain the following sufficient conditions for asymptotic stability in the large:

$$\hat{f}(x_1) > 0 \text{ for } x_1 \neq 0,$$

$$g(x_1) > 0 \text{ for } x_1 \neq 0,$$

$$\int_0^{x_1} x_1 g(x_1) dx_1 \rightarrow \infty \text{ for } |x_1| \rightarrow \infty.$$

Note, the above conditions do not demand that the instantaneous dissipation be positive, but only the "average" dissipation be positive. An example of this, consider

$$g(x_1) = 1,$$

$$f(x_1) = x_1^4 - 7 x_1^2 + 12,$$

then,

$$\hat{f}(x_1) = \frac{x_1^4}{5} - \frac{7}{3} x_1^2 + 12 > 0 \text{ for all real } x_1,$$

and

$$f(x_1) < 0 \text{ if } -2 < x_1 < -\sqrt{3} \text{ and } \sqrt{3} < x_1 < 2.$$

(41) Second-Order Nonlinear Feedback System

The feedback system is described by the matrix equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -g(x_1) & 1 \\ -h(x_1) & -f(x_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This equation can be reduced to a special case of the single second-order equation in example (39). But the difficulties in applying the previous results are that the derivative of $g(x_1)$ must exist, and the "dissipation" term, the coefficient of \dot{x}_1 , in the single second-order equation is not of constant sign in the vicinity of $x_1 = x_2 = 0$.

In the CLF - technique the corresponding linear equation becomes

$$\ddot{x} + (f + g) \dot{x} + (fg + h) x = 0,$$

$$x_1 = x, \quad x_2 = \dot{x} + gx,$$

f, g, h are constants.

The system is stable if

$$fg + h > 0,$$

$$f + g > 0.$$

The Q matrix in this case becomes

$$Q(f, g, h) = \begin{bmatrix} 2(P_{11}g + P_{12}h) & P_{22}h + P_{12}(f + g) - P_{11} \\ P_{22}h + P_{12}(f + g) - P_{11} & 2(P_{22}f - P_{12}) \end{bmatrix}$$

For the nonlinear case, the choice of P_{11}, P_{12} must be such that for constant f, g, h we get the above stability requirements. From these conditions, we see that if $g > 0$, then any f greater than some minimum, \underline{f} , is satisfactory.

Thus, the values of P_{ij} are taken as

$$P_{11} = h + \underline{f} (\underline{f} + g),$$

$$P_{12} = \underline{f},$$

$$P_{22} = 1,$$

and we have

$$Q = \begin{bmatrix} 2(h + \underline{f} g) & (\underline{f} + g) \\ \underline{f} (f - \underline{f}) & 2 (f - \underline{f}) \end{bmatrix}.$$

Therefore, Q is positive semi-definite if $(h + \underline{f}g)(\underline{f} + g) > \left[\frac{\underline{f}^2 (f - \underline{f})}{4} \right] \geq 0$.

For the nonlinear $f(x_1)$ we define \underline{f} as

$$\underline{f} = \min_{x_2} \left[f(x_2) \right],$$

thus $f \geq \underline{f}$. We also define \bar{f} as

$$\bar{f} = \max_{x_2} \left[f(x_2) \right].$$

For the nonlinear problem, we make use of the above CLF and obtain the following new Liapunov function:

$$V = 2 \int_0^{x_1} \left[\underline{f} g(x_1) + h(x_1) \right] x_1 dx_1 + (\underline{f} x_1 + x_2)^2,$$

where

$$\dot{V} = - \underline{x}_T Q \underline{x}.$$

The conditions for asymptotic stability in the large for this nonlinear system are

$$\underline{f} g(x_1) + h(x_1) > 0,$$

$$(\underline{f} + g(x_1))(\underline{f} g(x_1) + h(x_1)) > \frac{\underline{f}^2 (\bar{f} - \underline{f})}{4} \geq 0,$$

$$\int_0^{x_1} \left[\underline{f} g(x_1) + h(x_1) \right] x_1 dx_1 \rightarrow \infty \text{ as } |x_1| \rightarrow \infty.$$

An example of the above problem with non-differentiable functions is:

$$\begin{aligned} f(x_1) &= 2 + a e^{-|x_1|}, \quad a > 0, \\ g(x_1) &= -e^{-x_1^2}, \end{aligned}$$

and

$$h(x_1) = 6 + 2|x_1|.$$

Therefore, $\underline{f} = 2$ and $\bar{f} = 2 + a$. Now $(\underline{f} g + h) \geq -2 + 6 = 4 > 0$,

$\underline{f} + g \geq 2 - 1 = 1 > 0$, and $(\underline{f} g + h)(\underline{f} + g) \geq (4)(1) = 4$. Thus, we could let $\underline{f}^2 (\bar{f} - \underline{f}) / 4 = 3$, or $\underline{f} = 5$. Therefore for stability "a" can be in the interval $0 \leq a \leq 3$.

$$(42) \quad \ddot{\ddot{x}} + f(x, \dot{x}, \ddot{x}) \ddot{\ddot{x}} + g(x, \dot{x}) \dot{\ddot{x}} + h(x) x = 0.$$

The standard state variable form gives

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -h(x_1) x_1 - g(x_1, x_2) x_2 - f(x_1, x_2, x_3) x_3$$

where $x_1 = x$. When f, g, h are constants, the Routh-Hurwitz conditions for stability are

$$f, g, h > 0,$$

$$fg - h > 0.$$

This rather general third order, autonomous system will be analyzed by first considering several special cases.

(A) $\ddot{x} + f(\dot{x}) \ddot{x} + g(\dot{x}) \dot{x} + h x = 0, \quad h \equiv \text{constant}.$

Using the usual notation, the Q matrix is

$$Q = \begin{bmatrix} 2 P_{13} h & P_{13} g + P_{23} h - P_{11} & P_{33} h + P_{13} f - P_{12} \\ P_{13} g + P_{23} h - P_{11} & 2(P_{23} g - P_{12}) & P_{33} g + P_{23} f - P_{22} - P_{13} \\ P_{33} h + P_{13} f - P_{12} & P_{33} g + P_{23} f - P_{22} - P_{13} & 2(P_{33} f - P_{23}) \end{bmatrix}.$$

The choice which is made to simplify Q is to let all $q_{ij} = 0$, except q_{22} and q_{33} . The elements q_{22} and q_{33} are selected to be

$$q_{22} = 2 (mg - h),$$

$$q_{33} = 2 (f - m),$$

where m is a constant. The resulting P_{ij} 's which are obtained are

$$P_{13} = 0, \quad P_{33} = 1, \quad P_{23} = m, \quad P_{11} = mh, \quad P_{12} = h,$$

$$P_{22} = q + mf.$$

Since Q is positive semi-definite, it is necessary that $\underline{f} - m \geq 0$ and

$$m \underline{g} - h \geq 0, \text{ where}$$

$$\underline{f} = \min_{x_2} [f(x_2)],$$

and

$$\underline{g} = \min_{x_2} [g(x_2)],$$

and where $Q \neq 0$. The value of m is selected such that $(\underline{f} - m)(m \underline{g} - h)$ is maximized,

or

$$m = \frac{\underline{f} \underline{g} + h}{2 \underline{g}}.$$

Thus, Q is positive semi-definite if $\underline{f} \underline{g} > 0$ and

$$\underline{f} \underline{g} - h > 0.$$

Now using the P_{ij} 's and the Q matrix, we get the following Liapunov function for the nonlinear system:

$$V = m h x_1^2 + 2 h x_1 x_2 + 2 \int_0^{x_2} [g(x_2) + m f(x_2)] x_2 dx_2 + \\ + 2 m x_2 x_3,$$

and

$$V \geq \underline{x}_T \underline{P}^* \underline{x} = \underline{x}_T \underline{x},$$

mh	h	0
h	$\underline{g} + m \underline{f}$	m
0	m	1

where \underline{P}^* is positive definite. The system is asymptotically stable in the large if

$$\underline{f}, \underline{g}, h > 0,$$

$$\underline{f}, \underline{g} - h > 0.$$

(B) $\ddot{x} + f(\dot{x}) \dot{x} + g(x) \dot{x} + h(x) x = 0.$

Since we replaced h by $h(x_1)$ in this case, the change in the Liapunov function is to replace $h x_1^2$ by $2 \int_0^{x_1} h(x_1) x_1 dx_1$. The resulting Liapunov function takes on the form:

$$V = 2 m \int_0^{x_1} h(x_1) x_1 dx_1 + 2 h(x_1) x_1 x_2 + 2 m x_2 x_3 + x_3^2 + \\ + 2 \int_0^{x_2} [g(x_2) + m f(x_2)] x_2 dx_2.$$

The constant m in V must be redefined.

The time derivative of V gives

$$\dot{V} = - \underline{x}_T Q \underline{x} = - \underline{x}_T$$

0	0	0
0	$2(mg - H')$	0
0	0	$2(f - m)$

\underline{x} ,

where $H(x_1) = h(x_1) x_1$ and $H' = \frac{dH}{dx_1}$. Thus for $\underline{Q} \geq \underline{0}$, it is necessary that

$$\underline{f} - m \geq 0,$$

and

$$m \underline{g} - \overline{H'} \geq 0,$$

where

$$\overline{H'} = \max_{x_1} [H'(x_1)].$$

Let m be selected so as to maximize the product $(\underline{f} - m)(m \underline{g} - \overline{H'})$. Then,

$$m = \left[\frac{\underline{f} \underline{g} + \overline{H'}}{2 \underline{g}} \right].$$

Thus, the Liapunov Function for the nonlinear system is bounded from below by

$$V \geq 2m \int_0^{x_1} H(x_1) dx_1 + 2H(x_1) x_2^2 + \underline{g} x_2^2 + m \underline{f} x_2^2 + 2m x_2 x_3 + x_3^2.$$

To summarize, the system is asymptotically stable in the large if

$$h(x_1) > 0, x_1 \neq 0,$$

$$\underline{f}, \underline{g}, \overline{H'} > 0,$$

$$\underline{f} \underline{g} - \overline{H'} > 0,$$

and

$$\int_0^{x_1} H(x_1) dx_1 \longrightarrow \infty \quad \text{as} \quad |x_1| \longrightarrow \infty .$$

(c) The General Case

We want the general case to include the two special cases A and B. Thus, in the integral in V, the functions f and g are replaced by functions of only x_2 ; namely, $\hat{g}(x_2)$ and $\hat{f}(x_2)$. The new functions \hat{f} and \hat{g} must reduce to f and g in case B. Therefore, the V-function is

$$V = 2m \int_0^{x_1} H(x_1) dx_1 + 2H(x_1)x_2 + 2mx_2x_3 + x_3^2 + \\ + 2 \int_0^{x_2} \left[\hat{g}(x_2) + m \hat{f}(x_2) \right] x_2 dx_2,$$

where m, \hat{g} and \hat{f} are to be determined. The time derivative of \dot{V} becomes

$$\dot{V} = - \underline{x}^T \underline{Q} \underline{x} = - \underline{x}^T \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 2(m\hat{g} - H') & (g - \hat{g}) + m(f - \hat{f}) \\ \hline 0 & (g - \hat{g}) + m(f - \hat{f}) & 2(f - m) \\ \hline \end{array} \underline{x} ,$$

To insure that \underline{Q} is positive semi-definite, we demand that

$$\begin{aligned} m \hat{g} - H' &> 0 , \\ \hat{f} - m &> 0 , \end{aligned}$$

and

$$4(m \hat{g} - H')(\hat{f} - m) - \text{Max} \left[(g - \hat{g}) + m(f - \hat{f}) \right]^2 > 0.$$

The functions \hat{f} and \hat{g} are now chosen to minimize $\text{Max} \left[(g - \hat{g}) + m(f - \hat{f}) \right]^2$.

Since, the final conditions are given

First, the following definitions are given:

$$\begin{aligned}\bar{f}(x_2) &= \max_{x_1, x_3} [f(x_1, x_2, x_3)] , & \bar{g}(x_2) &= \max_{x_1} [g(x_1, x_2)] \\ \underline{f}(x_2) &= \min_{x_1, x_3} [f(x_1, x_2, x_3)] , & \underline{g}(x_2) &= \min_{x_1} [g(x_1, x_2)] \\ \hat{f}(x_2) &= \frac{\bar{f}(x_2) + \underline{f}(x_2)}{2} & \hat{g}(x_2) &= \frac{\bar{g}(x_2) + \underline{g}(x_2)}{2} \\ \tilde{f} &= \max_{x_2} \left[\frac{\bar{f}(x_2) - \underline{f}(x_2)}{2} \right] , & \tilde{g} &= \max_{x_2} \left[\frac{\bar{g}(x_2) - \underline{g}(x_2)}{2} \right] \\ \underline{\underline{f}} &= \min_{x_2} [\underline{f}(x_2)] , & \underline{\underline{g}} &= \min_{x_2} [\underline{g}(x_2)]\end{aligned}$$

Thus, the last condition which is stated above for a positive semi-definite Q becomes

$$q \equiv 4 \left(m \underline{\underline{g}} - \frac{\tilde{f}^2}{4} \right) (f - m) - (\tilde{g} + m \tilde{f})^2 > 0.$$

If now m is chosen to maximize q , we obtain

$$m = \frac{2 \underline{\underline{f}} \underline{\underline{g}} + 2 \frac{\tilde{f}^2}{4} - \tilde{f} \tilde{g}}{\tilde{f}^2 + 4 \underline{\underline{g}}}.$$

For this value of m , $q > 0$ if $\underline{\underline{f}} \underline{\underline{g}} - \frac{\tilde{f}^2}{4} > \in$

where

$$\in = \sqrt{\tilde{f}^2 \underline{\underline{f}} \underline{\underline{H}}' + \tilde{f} \tilde{g} (\underline{\underline{f}} \underline{\underline{g}} + \underline{\underline{H}}') + \tilde{g}^2 \underline{\underline{g}}}.$$

In summary, the general third order system is asymptotically stable in the large if:

$$h(x_1) > 0 \text{ for } x_1 \neq 0 ,$$

$$\underline{\underline{f}}, \underline{\underline{g}}, \underline{\underline{H}}' > 0 ,$$

$$\underline{f} \underline{g} - \overline{H'} > \epsilon \geq 0, \text{ and}$$

$$\int_0^{x_1} H(x_1) dx_1 \longrightarrow \infty \text{ as } |x_1| \longrightarrow \infty.$$

Special Cases of Part C

(1) Let f , g and h be constants. Then $\widetilde{f} = \widetilde{g} = \epsilon = 0$, $f = \underline{f}$, $g = \underline{g}$, $\overline{H'} = h$, and the Liapunov function is

$$V = \frac{h(fg + h)}{2g} x_1^2 + 2h x_1 x_2 + 2 \frac{(fg + h)}{2g} x_2 x_3$$

$$+ x_3^2 + 2g \frac{x_2^2}{2} + \frac{f(fg + h)}{2g} x_3^2.$$

The conditions for asymptotic stability are

$$h > 0, f > 0, g > 0 \text{ and}$$

$$fg - h > 0.$$

(2) Let f be a constant, $g = g(\dot{x})$ and $h = h(x)$. Then $\widetilde{f} = \widetilde{g} = \epsilon = 0$,

$\underline{f} = f$, and \underline{g} and $\overline{H'}$ are defined as before. Thus, the stability conditions are the same as Barbashin's results:

$$h(x_1) > 0, x_1 \neq 0$$

$$f, \underline{g}, \overline{H'} > 0$$

$$\underline{f} \underline{g} - \overline{H'} > 0$$

$$\int_0^{x_1} [H(x_1)] dx_1 \longrightarrow \infty \text{ as } |x_1| \longrightarrow \infty.$$

(3) Let $f = f(\dot{x})$ and $g = g(\dot{x})$. When h is a constant, we get case A.

When $h = h(x)$, we get case B.

(4) For a particular example, consider

$$f(x_1, x_2, x_3) = 1 + \frac{a}{|x_2| + 1} + ae^{-(x_1^2 + x_2^2 + x_3^2)},$$

$$g(x_1, x_2) = 6 + b e^{-(x_1^6 + x_2^2)} + \frac{2b}{x_2^4 + 2},$$

$$h(x_1) = 1 - \frac{3}{x_1} \left[1 - (1 + x_1)e^{-x_1} \right] \text{ for } x_1 > 0,$$

$$h(-x_1) = h(x_1).$$

We want to find the range of a and b so that stability is guaranteed. Thus, we obtain the following:

$$\bar{f}(x_2) = 1 + \frac{a}{|x_2| + 1} + ae^{-\frac{x_2^2}{2}}$$

$$\underline{f}(x_2) = 1$$

$$\hat{f}(x_2) = \frac{a}{2} \left[e^{-\frac{x_2^2}{2}} + \frac{1}{|x_2| + 1} \right] + 1$$

$$\bar{f} = \max_{x_2} \left\{ \frac{a}{2} \left[e^{-\frac{x_2^2}{2}} + \frac{1}{|x_2| + 1} \right] \right\} = a$$

$$f = 1$$

$$\bar{g}(x_2) = 6 + b e^{-x_2^2} + \frac{2b}{x_2^4 + 2}$$

$$g(x_2) = 6$$

$$\wedge g(x_2) = 6 + \frac{b}{2} \left[e^{-x_2^2} + \frac{2}{x_2^4 + 2} \right]$$

$$\tilde{g} = \max_{x_2} \frac{b}{2} \left[e^{-x_2^2} + \frac{2}{x_2^4 + 2} \right] = b$$

$$\underline{g} = 6$$

$$H(x_1) = x_1 - 3 \left[1 - (1 + x_1) e^{-x_1} \right], \quad x_1 > 0$$

$$H(-x_1) = -H(x_1)$$

$$H'(x_1) = 1 - 3x_1 e^{-x_1}, \quad x_1 > 0$$

$$H'(-x_1) = H'(x_1), \text{ and}$$

$$\overline{H'} = 1.$$

Therefore, $\underline{f} \underline{g} - \overline{H'} = 5 > 0$ and we can select $\epsilon < 5$. Then,

$$\epsilon = \sqrt{\tilde{f}^2 \underline{f} \overline{H'} + (\tilde{f} \tilde{g})(\underline{f} \underline{g} + \overline{H'}) + \tilde{g}^2 \underline{g}}$$

$$= \sqrt{a^2 + 7ab + 6b^2} < 5,$$

or

$$a^2 + 7ab + 6b^2 < 25.$$

The region of allowable a and b for asymptotic stability in the large is

$$a \geq 0, b \geq 0, a^2 + 7ab + 6b^2 < 25.$$

(D) $\ddot{x} + f\ddot{x} + g(x)\dot{x} + hx = 0.$

In this case, f and h are constants. Case D is a special case of C but if considered separately, less stringent conditions for stability can be derived. The state variable form of this equation is obtained by a logical extension of the Lienard transformation:

$$\dot{y}_1 = y_3$$

$$\dot{y}_2 = -h y_1$$

$$\dot{y}_3 = -G(y_1) + y_2 - f y_3$$

where

$$y_1 = x$$

and

$$G = G(y_1) = \int_0^{y_1} g(y_1) dy_1.$$

Assuming the usual forms of V and \dot{V} , we obtain for the Q -matrix the following:

$$Q = \begin{array}{|c|c|c|} \hline 2(h P_{12} + G/y_1 P_{13}) & hP_{22} + G/y_1 P_{23} - P_{13} & hP_{23} + \frac{G}{y_1} P_{33} + fP_{13} - P_{11} \\ \hline hP_{22} + G/y_1 P_{23} - P_{13} & -2 P_{23} & fP_{23} - P_{12} - P_{33} \\ \hline hP_{23} + G/y_1 P_{33} + fP_{13} - P_{11} & fP_{23} - P_{12} - P_{33} & 2(fP_{33} - P_{13}) \\ \hline \end{array}$$

We now select the elements of \underline{P} such that

$$Q = \begin{array}{|c|c|c|} \hline 2(f G/y_1 - h) & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}.$$

Thus

$$P_{11} = f^2 + G/y_1, \quad P_{12} = -1,$$

$$P_{22} = f/h, \quad P_{13} = f, \quad P_{33} = 1, \quad P_{23} = 0,$$

and, hence

$$V = f^2 y_1^2 + 2 \int_0^{y_1} G(y_1) dy_1 - 2y_1 y_2 + f/h y_2^2 + \\ + 2 f y_1 y_3 + y_3^2,$$

or

$$V = 2/f \int_0^{y_1} \left[f \frac{G(y_1)}{y_1} - h \right] y_1 dy_1 + \frac{f}{h} (y_2 - \frac{h}{f} y_1)^2 \\ + (f y_1 + y_3)^2,$$

and

$$\dot{V} = -2 (fG/y_1 - h) y_1^2.$$

In summary, the system is asymptotically stable in the large if

$$f > 0, \quad h > 0, \quad G(y_1)/y_1 > 0 \text{ for } y_1 \neq 0,$$

$$fG(y_1)/y_1 - h > 0,$$

$$\int_0^{y_1} (f G(y_1)/y_1 - h) y_1 dy_1 \longrightarrow \infty \text{ as } |y_1| \longrightarrow \infty.$$

In passing, we note that only $\left[\frac{G(y_1)}{y_1} \right]$, must be positive and not necessarily

$g(y_1)$. An example of this fact is:

$$f = 36, \quad h = 186,$$

$$g(y_1) = y_1^4 - 7 y_1^2 + 12$$

where

$$g(y_1) < 0 \text{ when } -2 < y_1 < -\sqrt{3} \text{ and } \sqrt{3} < y_1 < 2.$$

The corresponding $\left[\frac{G(y_1)}{y_1} \right]$ is $\left[\frac{y_1^4}{5} - \frac{7y_1^2}{3} + 12 \right]$, which is positive, and

$$\min \left\{ \frac{G(y_1)}{y_1} \right\} \text{ is } \frac{187}{36}.$$

Thus, $f \frac{G(y_1)}{y_1} - h > f \left(\frac{187}{36} \right) - h = 1 > 0$. Therefore, all stability conditions

are satisfied.

The next example is from a paper, [16], by DiStefano. The subject of the paper is concerned with the best choice of state variables for the stability studies of nonlinear systems by the Liapunov Method. The example is the same as one of Schultz and Gibson's, example (2) in this compendium. But by a different choice of state variables, DiStefano was able to relax some of the conditions on the non-linearities in the system.

(43) Second Order Example

The system equations using the canonical state variables as per Schultz and Gibson are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - x_2 f(x_1) - x_1 x_2 \frac{df(x_1)}{dx_1} - x_1 \beta f(x_1).$$

The resulting Liapunov function and its time derivative were found to be

$$V = 2 \int_0^{x_1} \left[\beta x_1 f(x_1) + x_1 \frac{dw}{dx_1} \right] dx_1 + (x_1 + x_2)^2,$$

$$\dot{V} = -2 x_2^2 \frac{dw}{dx_1} - 2 \beta x_1^2 f(x_1),$$

where $w = x_1 f(x_1)$. Thus, one of the restrictions on the nonlinearity is that w has positive slope.

By using "block diagrams" the author determined a new set of state variables which can be shown to be stable without restricting the derivative $\frac{dw}{dx_1}$. The resulting set of equations and the Liapunov function, which was

derived by the Variable Gradient Method, are

$$\dot{x}_1 = -x_1 f(x_1) + x_2$$

$$\dot{x}_2 = x_1 f(x_1) (1 - \beta) - x_2$$

$$V = x_2^2 + 2 (\beta - 1) \int_0^{x_1} x_1 f(x_1) dx_1$$

$$\dot{V} = -2 x_2^2 - 2 (\beta - 1) x_1^2 f^2(x_1).$$

Thus, for $V > 0$ and $\dot{V} < 0$, it is sufficient that $\beta > 1$ and $w = x_1 f(x_1) > 0$; but no condition is placed on the slope $\frac{dw}{dx_1}$.

The final example was taken from a speech [17] given by Dr. J. LaSalle in Iowa, in 1964.

(44) Third Order Example

The system equations in state variable form are

$$\dot{x}_1 = f(x_1, x_2)$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -p(x_1) - q(x_2) - a x_3.$$

The choice of \underline{V} for this system is

$$\underline{V} = \begin{bmatrix} \alpha(x_1) + \frac{d\beta(x_1)}{dx_1} x_2 \\ \beta(x_1) + \gamma(x_2) + 2cx_3 \\ 2cx_2 + 2x_3 \end{bmatrix},$$

where α , β and γ are undertermined functions and C is an unknown constant.

Therefore, the time derivative of V is given by

$$\begin{aligned} \dot{V} = (\underline{V})_T \dot{\underline{x}} = & \alpha f + \beta' f x_2 + \beta x_3 + \gamma x_3 + 2cx_3^2 + \\ & - 2cp x_2 - 2p x_3 - 2cq x_2 - 2q x_3 - 2ac x_2 x_3 - 2a x_3^2. \end{aligned}$$

Let

$$\gamma = 2ac x_2 + 2q(x_2)$$

$$\alpha = 2x_1$$

$$\beta = 2p(x_1), \text{ and } c = 1.$$

Then, \dot{V} becomes

$$\dot{V} = \begin{bmatrix} 2x_1 + p'(x_1)x_2 \\ 2p(x_1) + 2ax_2 + 2q(x_2) + 2x_3 \\ 2x_2 + 2x_3 \end{bmatrix}_T \underline{x},$$

or

$$\begin{aligned} \dot{V} = & -2 \left[p(x_1)x_2 + q(x_2)x_2 - \frac{dp(x_1)}{dx_1} f(x_1, x_2)x_2 \right] + 2x_1 f(x_1, x_2) + \\ & -2 \left[a - 1 \right] x_3^2. \end{aligned}$$

For the system (1) the time derivative of V is

We now check the curl equations:

$$\frac{\partial^2 V}{\partial x_1 \partial x_2} = 2 p(x) = \frac{\partial^2 V}{\partial x_2 \partial x_1},$$

$$\frac{\partial^2 V}{\partial x_3 \partial x_2} = 2 = \frac{\partial^2 V}{\partial x_2 \partial x_3},$$

$$\frac{\partial^2 V}{\partial x_1 \partial x_3} = 0 = \frac{\partial^2 V}{\partial x_3 \partial x_1}.$$

Thus, by line integration we can obtain V:

$$V = x_1^2 + 2 \int_0^{x_2} \left[p(x_1) + q(x_2) \right] x_2 dx_2 + a x_2^2 + 2 x_2 x_3 + x_3^2.$$

Therefore, the conditions of asymptotic stability are

$$x_1 f(x_1, x_2) < 0 \quad \text{for } x_1 \neq 0$$

$$p(x_1) x_2 + q(x_2) x_2 - \frac{d p(x_1)}{d x_1} f(x_1, x_2) x_2 > 0$$

$$a - 1 > 0$$

$$p(x_1) + q(x_2) > 0.$$

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SECTION SIX

LIAPUNOV FUNCTIONS

AND

AUTOMATIC CONTROL THEORY

Prepared by: H. Kwatny
J. Persechetti
G. Blundall

1. Some Definitions and Basic Theorems

In this section we set down some basic definitions and theorems which are used thruout the ensuing work. Thorough discussion and proofs are available in various text books and monographs including [1], [4] and [9]. We are primarily concerned with a system of autonomous differential equations of the general form.

$$\dot{\bar{x}} = \bar{f}(\bar{x}), \quad \bar{f}(0) = 0 \quad (1)$$

Let S_R denote the open sphere $\|\bar{x}\| < R$ and H_R its boundary and assume that (1) satisfies the existence and uniqueness theorem in a region S_A . Liapunov gives the following definition of stability of the origin.

Definition 1: The origin of (1) is stable if for any $S_\epsilon \subset S_A$ there is a $S_\delta(\epsilon) \subseteq S_\epsilon$ such that if $\bar{x}(t, \bar{x}_0)$ is the solution corresponding to an initial vector \bar{x}_0 which lies in S_δ , then $\bar{x}(t, \bar{x}_0)$ lies in S_ϵ thereafter.

Definition 2: The origin of (1) is asymptotically stable if the origin is stable and there is a $\delta_0 > 0$ such that $\bar{x}(t, \bar{x}_0) \rightarrow 0$ as $t \rightarrow +\infty$ for all \bar{x}_0 in S_{δ_0} .

Definition 3: The origin of (1) is unstable whenever it is not stable.

Definition 4: The origin of (1) is globally asymptotically stable if it is asymptotically stable and S_{δ_0} includes the whole state space.

In the sequel we will direct our attention to control systems which include one or more nonlinear functions $Q(\sigma)$. We will not in general specify $Q(\sigma)$ completely but will consider a class of functions which satisfy certain prescribed conditions. A function belonging to this class will be termed admissible.

Definition 5: The origin of the control system state space is absolutely stable if it is globally asymptotically stable for all admissible functions $Q(\sigma)$.

Theorem 1: (Liapunov Stability Theorem). Whenever there exists a positive definite function $V(\bar{x})$ in S_A whose derivative \dot{V} along the trajectories of (1) is negative semidefinite in S_A then the origin is stable.

Theorem 2: (Liapunov Asymptotic Stability Theorem). Whenever there exists a positive definite function $V(\bar{x})$ in S_A whose derivative \dot{V} along the trajectories of (1) is negative definite in S_A then the origin is asymptotically stable.

Theorem 3: (Liapunov Instability Theorem). If there exists a positive definite function $V(\bar{x})$ with continuous first partials in S_A and if $-\dot{V}$ is negative definite in S_A then the origin is unstable.

Theorem 4: (Barbashin-Krassovskii complement to Theorem 2) If the conditions of Theorem 2 hold and $S_A = S_\infty$ and $V \rightarrow \infty$ with $\|\bar{x}\|$, then all solutions tend to the origin.

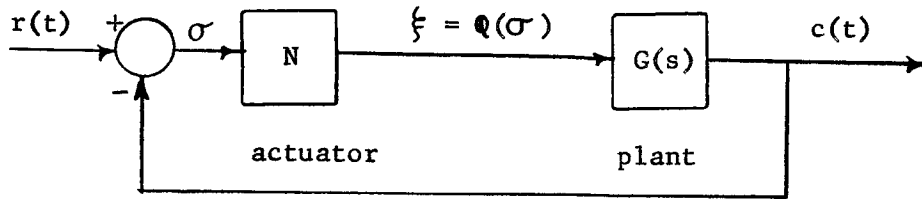
Theorem 5: (LaSalle, Barbashin and Krassovskii, Tuzov)

If (a) $V(\bar{x}) \rightarrow \infty$ with $\|\bar{x}\|$, (b) the locus of points such that $\dot{V} = 0$ contains no nontrivial solution, (c) V is positive definite and (d) $-\dot{V}$ is positive semidefinite then the origin is globally asymptotically stable.

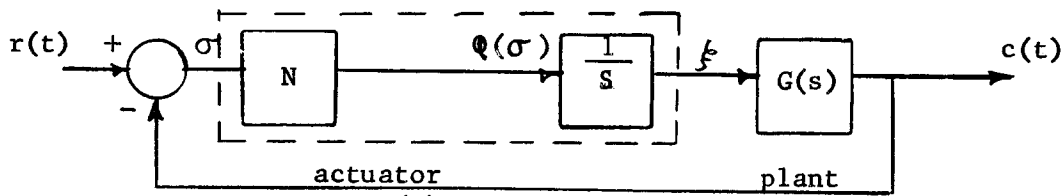
2. Equations of Motion

Much of the original work in the study of nonlinear control systems was based on a model composed of a linear plant and a nonlinear control element or actuator. If the actuator contained no dynamic characteristic, i.e., the feedback acted on the plant directly thru the nonlinearity, then the control was called 'direct'. On the other hand if the feedback acted thru one or more derivatives as well as the nonlinearity the control was termed 'indirect'.

This is illustrated in fig. (1). The system



(a)



(b)

fig(1)

in fig(1a) is a direct control system. That in fig(1b) is an example of an indirect control system. In present usage the term 'indirect' refers strictly to the configuration of fig(1b).

In general we will consider $r(t) = 0$ and it is clear that many control systems with a single nonlinearity can be put in the configuration of fig(1a). In particular, the system of fig(1b) is readily transformed into the configuration of fig(1a). However, the distinction between 'direct' and 'indirect' control systems is significant if we agree that $G(s)$ be stable, i.e., all poles of $G(s)$ have negative real parts. In this case the linear part of the indirect system has a pole at the origin and is unstable whereas the linear part of the direct system is stable. The treatment of the two situations will be different.

In more general cases of interest, however, $G(s)$ will have one or more poles at the origin and possibly poles elsewhere on the imaginary axis. Hence in more general situations the terminology of direct and indirect control loses its mathematical significance and it would be more convenient to classify the system according to the location of the roots of the characteristic equation of the linear part of the system. However the terminology still has some physical appeal and will be used.

If $G(s)$ is a real proper fraction in s with the order of the numerator less than or equal to that of the denominator its partial fraction expansion takes the form

$$G(s) = \alpha_0 + \frac{\alpha_1}{s - \lambda_1} + \dots + \frac{\alpha_n}{s - \lambda_n}$$

where n is the order of the denominator and λ 's which are not real occur in conjugate pairs. Then the system in fig(1b) can be redrawn as in fig(2).

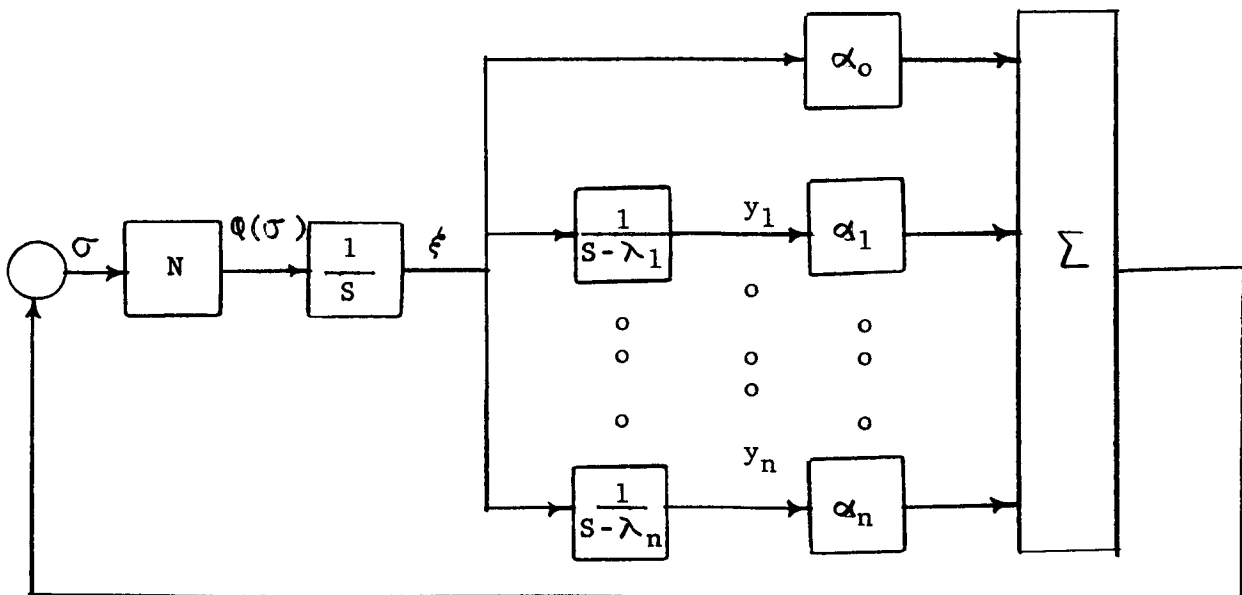


fig. (2)

The system in fig.(2) can be described by the system of equations

$$y_i = \lambda_i y_i + \xi \quad i = 1, 2, \dots$$

$$\sigma = \sum_{i=1}^n (-\alpha_i) y_i + (\alpha_0) \xi$$

$$\dot{\xi} = Q(\sigma)$$

or in matrix form

$$\left. \begin{aligned} \bar{y} &= A\bar{y} - \bar{b} \\ \sigma &= \bar{c}^t \bar{y} - \rho \xi \\ \dot{\xi} &= Q(\sigma) \end{aligned} \right\} \quad (1)$$

where

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \lambda_n \end{pmatrix}; \quad \bar{b} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ \vdots \\ -1 \end{pmatrix}$$

$$\bar{c}^t = (-1, -2, \dots, -n); \quad \rho = \alpha_0$$

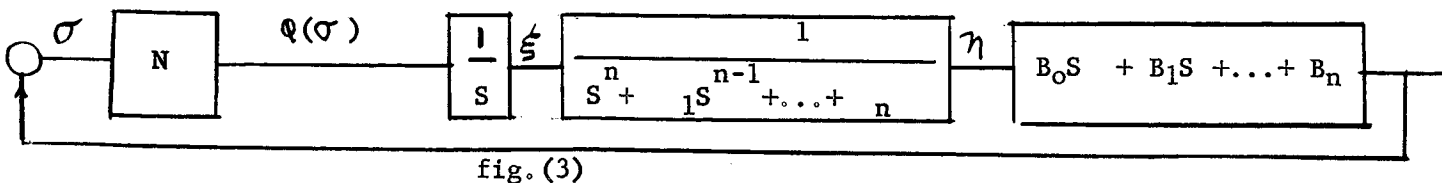
(Note: Since in general we are dealing with complex vectors and matrices the superscript t will designate the Hermitian transpose here)

This is Lur'e's canonic form.

The canonic form of the control system equations can be obtained in another way. Suppose the transfer function $G(s)$ is given as

$$G(s) = \frac{B_0 s^n + B_1 s^{n-1} + \dots + B_n}{s^n + \alpha_1 s^{n-1} + \dots + \alpha_n}$$

Then fig(1b) can be redrawn as in fig(3).



The set of equations describing this system is

$$\left. \begin{aligned} (D^n + \alpha_1 D^{n-1} + \dots + \alpha_n) \eta &= \xi \\ \dot{\xi} &= Q(\sigma) \\ \sigma &= -(B_0 D^n + B_1 D^{n-1} + \dots + B_n) \eta = (\gamma_1 D^{n-1} + \gamma_2 D^{n-2} + \dots + \gamma_n) \eta - \rho \xi \end{aligned} \right\} \quad (2)$$

where $D \equiv \frac{d}{dt}$,

$$\gamma_i = B_0 \alpha_i - B_i \quad i = 1, 2, \dots, n$$

$$\rho = B_0$$

Now the state space variables are defined by

$$(x_1, x_2, \dots, x_n) = (\eta, D\eta, \dots, D^{n-1}\eta)$$

so that the following first order system equivalent to (2) is obtained

$$D \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & & 0 & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & & -\alpha_2 & -\alpha_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \xi \end{pmatrix}$$

$$\begin{aligned} \dot{\xi} &= Q(\sigma) \\ &= \gamma_n x_1 + \dots + \gamma_1 x_n - \end{aligned}$$

or in vector notation

$$\left. \begin{aligned} \dot{\bar{x}} &= B \bar{x} - \bar{u} \\ \dot{\xi} &= Q(\sigma) \\ \sigma &= \bar{v}^t \bar{x} - s \end{aligned} \right\} \quad (3)$$

where

$$\bar{u} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix} ; \quad \bar{v} = \begin{pmatrix} n \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

This is sometimes referred to as the 'state' representation of the system.

The eigenvalues of the matrix B are the characteristic roots of the linear portion of the system. Let these be distinct and designated by $\lambda_1, \lambda_2, \dots, \lambda_n$. Then the canonic form is obtained by making use of the following Vandermonde matrix.

$$T = \begin{pmatrix} 1 & 1 & \dots & \dots & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \dots & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \dots & \dots & \lambda_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \dots & \dots & \lambda_n^{n-1} \end{pmatrix}$$

The canonic variables \bar{y} are defined by the transformation.

$$\bar{x} = T \bar{y}$$

Equations (3) become

$$\dot{\bar{y}} = T^{-1} B T \bar{y} - T^{-1} \bar{u}$$

$$\dot{\xi} = Q(\sigma)$$

or

$$\sigma = \bar{v}^t T \bar{y} - \rho \xi$$

$$\left. \begin{aligned} \dot{\bar{y}} &= A \bar{y} - \bar{b} \xi \\ \dot{\xi} &= Q(\sigma) \\ \sigma &= \bar{c}^t \bar{y} - \rho \xi \end{aligned} \right\}$$

(1)

where

$$A = T^{-1} B T = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

$$\bar{b} = T^{-1} u$$

$$\bar{c} = v^T T$$

In what follows the form of eq. (1) or (3) will be taken as representative of an indirect control system. However, it will not always be assumed that the eqs. are in canonic form. It will be stated when this assumption applies. The form for the direct control system is

$$\left. \begin{aligned} \bar{y} &= A \bar{y} - \bar{b} \xi \\ \xi &= Q(\sigma) \\ \sigma &= \bar{c}^T \bar{y} - \rho \xi \end{aligned} \right\} \quad (4)$$

If B_0 is zero, note that (4) becomes

$$\left. \begin{aligned} \bar{y} &= A \bar{y} - \bar{b} \xi \\ \xi &= Q(\sigma) \\ \sigma &= \bar{c}^T \bar{y} \end{aligned} \right\} \quad (5)$$

The 'characteristic' function $Q(\sigma)$ will be assumed, for the present, to have the following properties

1. $Q(\sigma)$ is defined and continuous for all σ
2. $Q(0) = 0$ and $\sigma Q(\sigma) > 0$ for all $\sigma \neq 0$
3. The integrals

$$\int_0^{\pm \infty} Q(\sigma) d\sigma \quad \text{diverge}$$

Any function having these properties is termed admissible. At times it will be necessary to modify the requirements on the class of admissible functions.

3. The Problem of Lur'e - Indirect Control

Initially Lur'e and Postnikov obtained sufficient conditions for the absolute stability of a class of direct and indirect control problems. The treatment given here follows a reformulation of the problem due to Lefshetz.

The indirect control system eqs. are

$$\begin{aligned}\dot{\bar{y}} &= A \bar{y} - \bar{b} \xi \\ \dot{\xi} &= Q(\sigma) \\ \sigma &= \bar{c}^t \bar{y} - \rho \xi\end{aligned}$$

It is convenient to transform from the variables (\bar{y}, ξ) to new variables (\bar{x}, σ) defined by the transformation

$$\left. \begin{aligned}\bar{x} &= A \bar{y} - \bar{b} \xi \\ \sigma &= \bar{c}^t \bar{y} - \rho \xi\end{aligned} \right\} \quad (1)$$

From (1) if \bar{y} and $\xi \rightarrow 0$ so does \bar{x} and σ . If the converse is true then (\bar{x}, σ) stability describes the (\bar{y}, ξ) stability. For this to be true (1) must have a unique inverse, i.e., we require

$$\begin{vmatrix} A & \bar{b} \\ \bar{c}^t & \rho \end{vmatrix} \neq 0$$

Now

$$\begin{vmatrix} A & \bar{b} \\ -\bar{c}^t & \rho \end{vmatrix} = |A| (\rho - \bar{c}^t A^{-1} \bar{b})$$

Since A is stable we know that $|A| \neq 0$. Hence we must have

$$\rho \neq \bar{c}^t A^{-1} \bar{b} \quad (2)$$

This relation between parameters of the system is assumed to be true. The system equations become

$$\begin{aligned}\dot{\bar{x}} &= A \bar{x} - B Q(\sigma) \\ \dot{\sigma} &= \frac{1}{c} \bar{x} - \rho Q(\sigma)\end{aligned}\quad (3)$$

If eq. (2) holds the only singular point of (3) is $\bar{x} = 0, \sigma = 0$ since $Q(\sigma) = 0$ if and only if $\sigma = 0$. This is a necessary condition for absolute stability.

The method used to determine the sufficient conditions for absolute stability is based upon Liapunov's asymptotic stability theorem and the Barbashin-Krassovskii complement. Lur'e and Postnikov considered a Liapunov function of the form

$$V(\bar{x}, \sigma) = \bar{x}^t B \bar{x} + \Phi(\sigma); \quad \Phi(\sigma) = \int_0^\sigma Q(\sigma) d\sigma \quad (4)$$

where B is a positive definite hermitian matrix. Recalling that $\sigma Q(\sigma) > 0$, $\sigma \neq 0$ and $Q(\sigma) = 0$ for $\sigma = 0$ it is seen that $V(\bar{x}, \sigma)$ is positive definite with respect to (\bar{x}, σ) . Furthermore recalling the requirement that $\int_0^\sigma Q(\sigma) d\sigma$ diverges as $\sigma \rightarrow \infty$ it follows that $V \rightarrow \infty$ for $\|(\bar{x}, \sigma)\| \rightarrow \infty$ and hence the Barbashin-Krassovskii complement is satisfied.

Differentiating (4) with respect to time

$$\dot{V}(\bar{x}, \sigma) = \dot{\bar{x}}^t B \bar{x} + \bar{x}^t B \dot{\bar{x}} + Q(\sigma) \dot{\sigma} \quad (5)$$

using (3)

$$\begin{aligned}\dot{V} &= \bar{x}^t A^t B \bar{x} - \bar{x}^t B A \bar{x} + \bar{x}^t B A \bar{x} - \bar{x}^t B B Q \\ &\quad + Q \frac{1}{c} \bar{x} - \rho Q^2 \\ \dot{V} &= \bar{x}^t (-A^t B - B A) \bar{x} + Q \left(\bar{x}^t B - \frac{1}{2} \bar{x}^t \right) \bar{x} + \bar{x}^t (B \bar{x} - \frac{1}{2} \bar{x}) \\ &\quad + \rho Q^2\end{aligned}$$

$$-\dot{V} = \bar{x}^t C \bar{x} + \rho Q Q^* + (Q^* \bar{d}^t \bar{x} + \bar{x}^t - Q) \quad (6)$$

Where

$$-C = A^t B + B A \quad (7)$$

$$= B \bar{b} - \frac{1}{2} \bar{c} \quad (8)$$

and use has been made of the fact that $Q = Q^*$ Eq. (6) can be written more compactly as

$$-V(x, \sigma) = (\bar{x}^t, Q^*) \begin{pmatrix} C & \bar{d} \\ \bar{d}^t & \rho \end{pmatrix} \begin{pmatrix} \bar{x} \\ Q \end{pmatrix} \quad (9)$$

According to a theorem by Sylvester necessary and sufficient conditions that $-\dot{V}$ be positive definite is that all of the principle minors of the matrix in (9) be positive. Hence we require that C be positive definite hermitian as well as

$$\begin{vmatrix} C & \bar{d} \\ \bar{d}^t & \rho \end{vmatrix} > 0$$

Now

$$\begin{vmatrix} C & \bar{d} \\ \bar{d}^t & \rho \end{vmatrix} = |C| (\rho - \bar{d}^t C^{-1} \bar{d}) > 0$$

Since $|C| \neq 0$ we require

$$\rho > \bar{d}^t C^{-1} \bar{d} \quad (10)$$

There are some comments to be made concerning the relationship between the matrices B and C. Eq. (7) is referred to in the literature as Liapunov's matrix equation. It can be shown (see [1] and [2]) that if the matrix A is stable and C is any given positive definite hermitian matrix then (7) has a unique solution B and B is a positive definite hermitian matrix. In particular if $A = \text{diag} (\lambda_1, \dots, \lambda_n)$ and $\text{Re } \lambda_i = -M_i < 0$ we have from (7)

$$(\lambda_j^* + \lambda_k) b_{jk} = -c_{jk}$$

$$b_{jk} = \frac{-c_{jk}}{\lambda_j^* + \lambda_k} \quad (11)$$

Hence, we have the following theorem.

Theorem: Sufficient conditions that the system of eq (1) be absolutely stable and that C be positive definite and the inequality (10) be satisfied.

The above formulation yields more general results than those originally obtained by Lur'e. For the purpose of comparison an analysis following that of Lur'e is given. Lur'e considered a special canonical system where

$$A = \text{diag}(A_1, \dots, A_n)$$

$$\bar{b}^t = (-1, \dots, -1)$$

$$\bar{c}^t = (-\alpha_1, \dots, -\alpha_n)$$

The Liapunov function has the form of (4) with C taken as

$$C = \bar{a} \bar{a}^t + \text{diag}(A_1, \dots, A_n)$$

and hence

$$B = -\frac{\lambda_j^* \alpha_k}{\lambda_j^* + \lambda_k} + \text{diag} \left(\frac{A_1}{2M_1}, \dots, \frac{A_n}{2M_n} \right) \text{ where } -M_i = \text{Re } \lambda_i < 0$$

To see that C is positive definite note that

$$\begin{aligned} \bar{x}^t C \bar{x} &= \bar{x}^t \bar{a} \bar{a}^t \bar{x} + \bar{x}^t \text{diag}(A_1, \dots, A_n) \bar{x} \\ &= |\bar{a}^t \bar{x}|^2 + \sum_{i=1}^n A_i |x_i|^2 \geq 0 \quad \text{for } \bar{x} \neq 0 \end{aligned}$$

if $A_i \geq 0$.

Now

$$\begin{aligned} -\dot{V} &= \bar{x}^t \text{diag}(A_1, \dots, A_n) \bar{x} \\ &+ \bar{x}^t \bar{a} \bar{a}^t \bar{x} + (\bar{c}^t \bar{a}^t \bar{x} + \bar{x}^t \bar{a} \bar{c}) + \rho \bar{c} \bar{c}^t \end{aligned}$$

Add and subtract $\sqrt{\rho} \mathbf{Q} (\bar{\mathbf{x}}^t \bar{\mathbf{a}} + \bar{\mathbf{a}}^t \bar{\mathbf{x}})$ to $-\dot{\mathbf{V}}$. Then

$$\begin{aligned} -\dot{\mathbf{V}} &= \bar{\mathbf{x}}^t \text{diag}(A_1, \dots, A_n) \bar{\mathbf{x}} + |\bar{\mathbf{a}}^t \bar{\mathbf{x}} + \sqrt{\rho} \mathbf{Q}|^2 \\ &\quad + (\mathbf{Q}^* \bar{\mathbf{a}}^t \bar{\mathbf{x}} + \bar{\mathbf{x}}^t \bar{\mathbf{a}} \mathbf{Q}) - \sqrt{\rho} (\bar{\mathbf{x}}^t \bar{\mathbf{a}} + \bar{\mathbf{a}}^t \bar{\mathbf{x}}) \\ &= \bar{\mathbf{x}}^t \text{diag}(A_1, \dots, A_n) \bar{\mathbf{x}} + |\bar{\mathbf{a}}^t \bar{\mathbf{x}} + \sqrt{\rho} \mathbf{Q}|^2 \\ &\quad + \mathbf{Q}(\bar{\mathbf{a}} - \sqrt{\rho} \bar{\mathbf{a}})^t \bar{\mathbf{x}} + \mathbf{Q} \bar{\mathbf{x}}^t (\bar{\mathbf{a}} - \sqrt{\rho} \bar{\mathbf{a}}) \end{aligned}$$

Since $A_i \geq 0$ we have at least a positive semi-definite form for $-\dot{\mathbf{V}}$ if we require

$$\bar{\mathbf{a}} - \sqrt{\rho} \bar{\mathbf{a}} = 0 \quad (12)$$

Eq(12) is equivalent to the n scalar equations

$$a_k \sum_{j=1}^n \frac{a_{j*}}{\lambda_j^* + \lambda_k} - \frac{A_k}{2M_k} + \frac{\alpha_k}{2} - \sqrt{\rho} a_k = 0, \quad K=1, 2, \dots, n \quad (13)$$

or since the only restriction on the A_i is that they be greater or equal to zero we have the requirements on k

$$a_k \sum_{j=1}^n \frac{a_{j*}}{\lambda_j^* + \lambda_k} + \frac{\alpha_k}{2} - a_k \sqrt{\rho} \geq 0, \quad K=1, 2, \dots, n \quad (14)$$

Equations (13) are called the 'prelimit' equations. The equalities of (14) are called the 'limit' equations. If we require that the inequalities of (14) hold this corresponds to $A_i > 0$ which gives a positive definite form to $-\dot{\mathbf{V}}$. If we allow the equalities then we have some $A_i = 0$ and $-\dot{\mathbf{V}}$ is only positive semi-definite.

We arrive at Lure's sufficient conditions for absolute stability by the following two considerations. We assume that for some classes of systems under consideration here that whenever the limit equations have a suitable solution then the prelimit equations have a suitable solution. Hence we reduce the complexity of the problem by examining the limit system. Also, Lur'e postulated that the a_i are real or occur in complex conjugate pairs corresponding to real or complex conjugate pair i (Note: It is readily shown that this insures that the left side of (14) is real). Hence we have

Lure's Theorem: A system described by the equations

$$\begin{aligned}\dot{x}_i &= \lambda_i x_i + Q(\sigma) & i = 1, 2, \dots, n & \quad \operatorname{Re}(\lambda_i) < 0 \\ \dot{\sigma} &= \sum_i -\alpha_i - \rho Q(\sigma)\end{aligned}$$

where complex λ_i and α_i occur in corresponding complex conjugate pairs is absolutely stable if there exists a set of roots a_1, a_2, \dots , satisfying the limit equalities of (14) such that the a_i are real or occur in complex conjugate pairs corresponding to λ_i .

Conditions comparable to (13) or (14) are easily obtained from inequality (10) of the Lefschetz formulation, $-\dot{V}$ is positive definite if we choose

$$\bar{d} = B \bar{b} - \frac{1}{2} \bar{c} = 0$$

Assuming the canonical form of the system equations and the above B

$$a_j = a_j^* \sum_{k=1}^n \frac{a_k}{\lambda_j^* + \lambda_k} - \frac{A_j}{2M_j} + \frac{\alpha_j}{2} = 0 \quad j=1, 2, \dots, n$$

or taking the complex conjugate and interchanging subscripts

$$a_k \sum_{j=1}^n \frac{a_j}{\lambda_j^* + \lambda_k} + \frac{\alpha_k}{2} - \frac{A_k}{2} = 0 \quad k=1, 2, \dots, n \quad (15)$$

Note that these eqs. are identical to (13) except for the $\sqrt{\rho}$ term.

4, The Problem of Lur'e - Direct Control

We consider here a direct control system described by the system of eqs.

$$\begin{aligned}\dot{\bar{x}} &= A \bar{x} - B Q(\sigma) \\ \sigma &= \bar{c}^t \bar{x}\end{aligned} \quad (1)$$

where A is a stable matrix and we assume the only singular point of (1) occurs at the origin. Differentiating the second equation with respect to time

$$\dot{\sigma} = \bar{c}^t \dot{\bar{x}} = \bar{c}^t A \bar{x} - \bar{c}^t B Q(\sigma)$$

Following Lefschetz we define

$$\bar{c}_0 = A^t \bar{c}, \quad \rho_0 = \bar{c}^t \bar{b} \quad (2)$$

so that

$$\dot{\sigma} = \bar{c}_0^t \bar{x} - \rho_0 \Phi(\sigma) \quad (3)$$

Taking the same Liapunov function as for the indirect control case

$$V(\bar{x}) = \bar{x}^t B \bar{x} + \Phi(\sigma) \quad (4)$$

Differentiating with respect to time

$$\begin{aligned} \dot{V} &= \bar{x}^t C \bar{x} + \bar{c}^t \bar{b} Q Q^* + (Q^* \bar{d}^t \bar{x} + \bar{x}^t \bar{d} Q) \\ A^t B + B A &= -C \end{aligned} \quad (5)$$

$$- = B \bar{b} - \frac{1}{2} A^t \bar{c}$$

and it appears that once again we have a hermitian form for \dot{V} in the $n + 1$ dimensional space (\bar{x}, Q)

which can be made positive definite by requiring C be positive definite and

$$\bar{c}^t \bar{b} - \bar{d}^t C^{-1} \bar{d}$$

as was done previously. However, this is not the case as pointed out by

Rosenbasser [3]. In fact \dot{V} can be at most positive semi-definite in the

$n + 1$ independent variables (\bar{x}, Q) . To see this note that \dot{V} can be written using

(1) and (4)

$$\begin{aligned} \dot{V} &= \bar{x}^t B \bar{x} + \bar{x}^t B \bar{x} + Q \bar{c}^t \bar{x} \\ &= \left[(\bar{x}^t B + \frac{1}{2} \bar{c}^t Q) \bar{x} \right]^* + \left[(\bar{x}^t B + \frac{1}{2} \bar{c}^t Q) \bar{x} \right] \\ &= \text{Re} \left[(2B \bar{x} + \bar{c} Q)^t (A \bar{x} - B Q) \right] \end{aligned} \quad (6)$$

Where use has been made of the fact that $\bar{c}^t \dot{x} = \dot{x}^t \bar{c}$ since both \dot{x} and \bar{c} occur in corresponding complex conjugate pairs. Now since A is a stable matrix we have $|A| \neq 0$ and hence there is a one dimensional subspace of the space (\bar{x}, Q) satisfying $A \bar{x} - \bar{b} Q = 0$. This means that \dot{V} can be zero at points other than the origin of (\bar{x}, Q) space and is therefore not definite in (\bar{x}, Q) space. Note that in order to have absolute stability of the system the only critical point of (1) can be the origin. Hence we must require that the only solution of $A \bar{x} - \bar{b} Q = 0$ is at the origin of (\bar{x}, Q) . If this is the case then eq (6) does not restrict \dot{V} to semi-definiteness. It seems plausible to inquire whether this requirement is sufficient to yield a definite \dot{V} . Lefshetz [2] shows that this is the case and focuses attention on what conditions guarantee that the only solution of $A \bar{x} - \bar{b} Q = 0$ is the trivial solution. He obtains the requirement that $\bar{c}^t A^{-1} \bar{b} \leq 0$ and proves the following theorem.

Theorem If $\bar{c}^t \bar{b} = \bar{d}^t C^{-1} \bar{d}$, C is positive definite and $\bar{c}^t A^{-1} \bar{b} \leq 0$, then the system (1) is absolutely stable.

The treatment given below follows that of Aizerman and Gantmacher 4. In order to avoid the difficulty discussed above add and subtract $\sigma Q(\sigma)$ to (5) then

$$-\dot{V} = S(x, Q) + \sigma Q(\sigma) \quad (7)$$

Where

$$S(\bar{x}, Q) = \bar{x}^t C \bar{x} + Q^* \left(\bar{d}^t - \frac{1}{2} \bar{c}^t \right) \bar{x} + \bar{x}^t \left(\bar{d} - \frac{1}{2} \bar{c} \right) Q + \bar{c}^t \bar{b} Q Q^* \quad (8)$$

Where use has been made of the facts $Q = Q^*$ and $\bar{c}^t \bar{x} = \bar{x}^t \bar{c}$. It is clear that \dot{V} will be negative definite if $S(\bar{x}, Q)$ is positive definite. Furthermore requiring $S(\bar{x}, Q)$ be positive definite as a hermitian form in the $N + 1$ variables (\bar{x}, Q) does not lead to a contradiction. Following closely to reference [4] we consider two cases separately. These being $r = \bar{c}^t \bar{b} > 0$ and $\bar{r} = \bar{c}^t \bar{b} = 0$.

$$\underline{r = \bar{c}^t \bar{b} > 0}$$

1st Approach: Note that equation (8) can be written

$$S(\bar{x}, Q) = (\bar{x}, Q) \begin{pmatrix} C & (\bar{d} - \frac{1}{2} \bar{c}) \\ (\bar{d} - \frac{1}{2} \bar{c})^t & \bar{c}^t \bar{b} \end{pmatrix} \begin{pmatrix} \bar{x} \\ Q \end{pmatrix}$$

Which is positive definite if C is positive definite and the Lefschetz inequality

$$\bar{c}^t \bar{b} > (\bar{d} - \frac{1}{2} \bar{c})^t C^{-1} (\bar{d} - \frac{1}{2} \bar{c}) \quad (9)$$

holds.

The simplest way to insure (9) is to require $\bar{d} - \frac{1}{2} \bar{c} = 0$.

2nd Approach: Noting that $\bar{c}^t \bar{b} > 0$ add and subtract the quantity

$$\frac{1}{\bar{c}^t \bar{b}} \left| (\bar{d} - \frac{1}{2} \bar{c})^t \bar{x} \right|^2 = \frac{1}{\bar{c}^t \bar{b}} \bar{x}^t (\bar{d} - \frac{1}{2} \bar{c}) (\bar{d} - \frac{1}{2} \bar{c})^t \bar{x}$$

to $S(x, Q)$

$$S(x, Q) = \left| \sqrt{\bar{c}^t \bar{b}} Q + \frac{1}{\bar{c}^t \bar{b}} (\bar{d} - \frac{1}{2} \bar{c})^t \bar{x} \right|^2 + \bar{x}^t C \bar{x} - \frac{1}{\bar{c}^t \bar{b}} \bar{x}^t (\bar{d} - \frac{1}{2} \bar{c}) (\bar{d} - \frac{1}{2} \bar{c})^t \bar{x} \quad (10)$$

In order that $S(x, Q)$ be positive definite it is necessary and sufficient that

$$Q(x) = \bar{x}^t C \bar{x} - \frac{1}{\bar{c}^t \bar{b}} \bar{x}^t (\bar{d} - \frac{1}{2} \bar{c}) (\bar{d} - \frac{1}{2} \bar{c})^t \bar{x} \quad (11)$$

be positive definite.

Now consider a system in Lure's first canonical form, i. e.,

$$A = \text{diag}(\lambda_1, \dots, \lambda_n)$$

$$\bar{c} = \begin{pmatrix} -\alpha_1^* \\ -\alpha_2^* \\ \vdots \\ -\alpha_n^* \end{pmatrix} \quad \bar{b} = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

and choose

$$C = \bar{a} \bar{a}^t + \text{diag}(A_1, \dots, A_n)$$

$$\text{then } B = - \frac{\bar{a}_j \bar{a}_k^*}{\lambda_j^* + \lambda_k} + \text{diag} \left(\frac{A_1}{2M_1}, \dots, \frac{A_n}{2M_n} \right) \quad \text{where } M_i = \text{Re } \lambda_i < 0$$

and

$$\begin{aligned} \left(\bar{d} - \frac{1}{2} \bar{c} \right)_k &= \left(B \bar{b} - \frac{1}{2} A^t \bar{c} - \frac{1}{2} \bar{c} \right)_k \\ &= \sum_{j=1}^n \frac{\bar{a}_j \bar{a}_k^*}{\lambda_j^* + \lambda_k} - \frac{A_k}{2M_k} + \frac{\lambda_k \alpha_k^*}{2} + \frac{\alpha_k^*}{2} \end{aligned}$$

Eq(11) can be written

$$\begin{aligned} Q(x) &= \bar{x}^t \text{diag}(A_1, \dots, A_n) \bar{x} \\ &+ \bar{x}^t \left[\bar{a} \bar{a}^t - \frac{1}{\bar{c}^t \bar{b}} \left(\bar{d} - \frac{1}{2} \bar{c} \right) \left(\bar{d} - \frac{1}{2} \bar{c} \right)^t \right] \bar{x} \quad (12) \end{aligned}$$

With $A_i > 0$ we can insure a positive definite $Q(x)$ by choosing the elements of \bar{a} so that

$$\bar{a} = \frac{1}{\sqrt{\bar{c}^t \bar{b}}} \left(\bar{d} - \frac{1}{2} \bar{c} \right) \quad (13)$$

Hence we have the n equations

$$\sqrt{\frac{t}{c}} \bar{b} a_k = \sum_{j=1}^n \frac{a_j^* a_k}{\lambda_j^* + \lambda_k} - \frac{A_k}{2M_k} + \frac{\lambda_k \alpha_k^*}{2} + \frac{\alpha_k^*}{2}, \quad k = 1, 2, \dots, n$$

or since A_k is an arbitrary positive constant we require only that the a_k satisfy

$$a_k \sum_{j=1}^n \frac{a_j^*}{\lambda_j^* + \lambda_k} + \frac{\lambda_k \alpha_k^*}{2} + \frac{\alpha_k^*}{2} - \sqrt{\frac{t}{c}} \bar{b} a_k > 0 \quad (14)$$

These are 'Lur'e's resolving prelimit equations'.

$$\underline{r = \frac{t}{c} \bar{b} = 0}$$

In this case eq (8) becomes

$$S(\bar{x}, Q) = \bar{x}^t C \bar{x} + \left[Q^* \left(\bar{a} - \frac{1}{2} \bar{c} \right)^t \bar{x} + \bar{x}^t \left(\bar{a} - \frac{1}{2} \bar{c} \right) Q \right]$$

which is positive definite in (\bar{x}, Q) if C is positive definite and we require

$$\bar{a} - \frac{1}{2} \bar{c} = 0 \quad (15)$$

Returning to the case of a system in the Lur'e first canonical form (15) leads to

$$\sum_{j=1}^n \frac{a_j^* a_k}{\lambda_j^* + \lambda_k} - \frac{A_k}{2M_k} + \frac{\lambda_k \alpha_k^*}{2} + \frac{\alpha_k^*}{2} = 0 \quad K = 1, 2, \dots, n$$

or

$$a_k \sum_{j=1}^n \frac{a_j^*}{\lambda_j^* + \lambda_k} + \frac{\lambda_k \alpha_k^*}{2} + \frac{\alpha_k^*}{2} > 0, \quad k = 1, 2, \dots, n \quad (16)$$

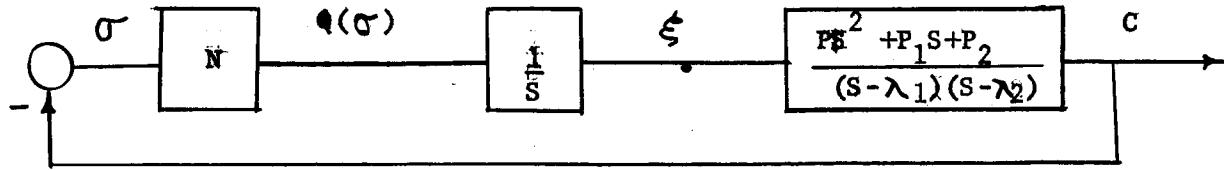
These are the 'prelimit' equations for the $\frac{t}{c} \bar{b} = 0$ case. Note eqs. (16) are contained in (14) although the two systems are obtained in different fashions.

Hence we can refer to (14) as the 'prelimit' equations in both cases. The so-called 'limit' equations are

$$a_k \sum_{j=1}^n \frac{a_j^*}{\lambda_j^* + \lambda_k} + \frac{\lambda_k \alpha_k^*}{2} + \frac{\alpha_k^*}{2} - \sqrt{\frac{t}{c}} \bar{b} a_k = 0 \quad (17)$$

$k = 1, 2, \dots, n$

Example: [2] We consider here the problem of an indirect control system with a second order plant. The control system under consideration is illustrated in fig.(1).

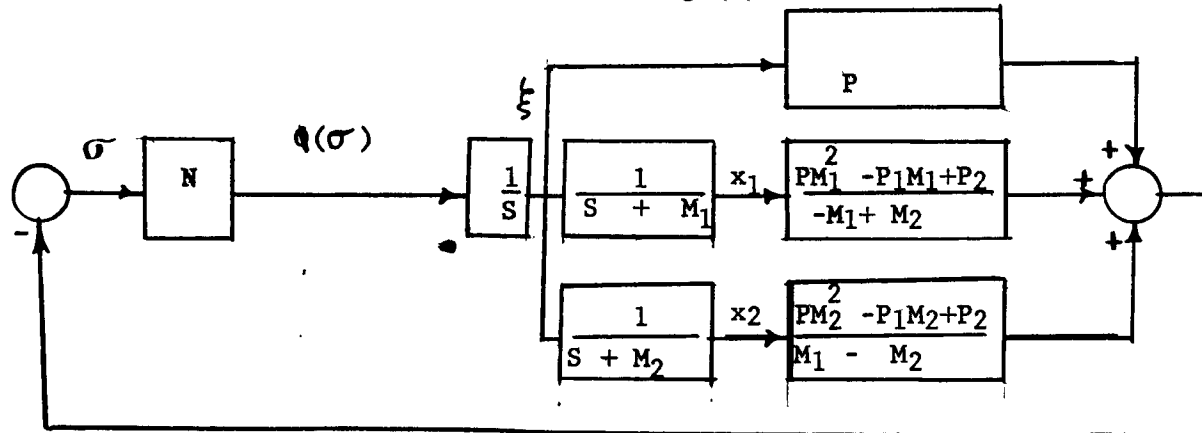


fig(1)

Three cases are considered; λ_1, λ_2 real and distinct, λ_1, λ_2 real and equal, λ_1, λ_2 complex.

CASE I - real roots; $\lambda_1 = -M_1 < 0, \lambda_2 = -M_2 < 0$

The diagram of fig(1) can be redrawn as shown in fig.(2)



fig(2)

The equations of motion are

$$\left. \begin{aligned} \dot{x}_1 &= -M_1 x_1 + \xi \\ \dot{x}_2 &= -M_2 x_2 + \xi \\ \sigma &= c_1 x_1 + c_2 x_2 - \rho \xi \end{aligned} \right\} \quad (1)$$

Where

$$C_1 = \frac{PM_1^2 - P_1M_1 + P_2}{M_1 - M_2}$$

$$C_2 = \frac{PM_2^2 - P_1M_2 + P_2}{-M_1 + M_2}$$

Choose

$$C = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$$

Since we require $C > 0$ we must have

$$p > 0, \quad pr - q^2 > 0 \quad (2)$$

Then we know

$$B = \begin{pmatrix} p_0 & q_0 \\ q_0 & r_0 \end{pmatrix} > 0$$

Where

$$p_0 = \frac{p}{2M_1}, \quad q_0 = \frac{q}{M_1 + M_2}, \quad r_0 = \frac{r}{2M_2} \quad (3)$$

As sufficient conditions for absolute stability we shall require

$$\bar{d} = B \bar{b} - \frac{1}{2} \bar{c} = 0 \quad (4)$$

Note: We have expressly indicated that (4) is a sufficient condition if

$p > 0$. This is also true if $p = 0$.

Eq.(4) yields

$$p_0 + q_0 + \frac{1}{2} c_1 = 0 \quad (5)$$

$$q_0 + r_0 + \frac{1}{2} c_2 = 0$$

Inequalities (2) are equivalent to

$$p_0 > 0, \quad p_0 r_0 - \frac{1}{4} q_0^2 > 0 \quad (6)$$

Where

$$\epsilon = \frac{(M_1 + M_2)^2}{4M_1 M_2} > 1$$

Using (5) to eliminate p_0 and r_0 from (6) we obtain

$$g(q_0) = (\epsilon - 1) q_0^2 - \frac{1}{2} (c_1 + c_2) q_0 - \frac{1}{4} c_1 c_2 < 0 \quad (7a)$$

$$q_0 < -\frac{1}{2} c_1 \quad (7b)$$

The discriminant of $g(q_0)$ is

$$\delta = \frac{(c_1 - c_2)^2}{4} + 4 c_1 c_2$$

Since $\epsilon > 0$, the parabola $g(q_0) = 0$ is concave upward. Hence if $\delta \leq 0$ inequality (7a) has no solution and we cannot demonstrate absolute stability by this method. If $\delta > 0$ then $g(q_0) = 0$ has two distinct real roots q_1, q_2 and we can satisfy (7a) by selecting q_0 in the interval (q_1, q_2) . Inequality (7b) must also be satisfied. Note that

$$g(-\frac{1}{2} c_1) = \epsilon \frac{c_1^2}{4} > 0$$

Then $-\frac{1}{2} c_1$ is not in the interval (q_1, q_2) . Hence $q_0 < -\frac{1}{2} c_1$ can be fulfilled only if $\frac{1}{2} (q_1 + q_2) < -\frac{1}{2} c_1$, or

$$-\frac{1}{2} c_1 > \frac{c_1 + c_2}{4(\epsilon - 1)}$$

or finally

$$(2\epsilon - 1) c_1 < -c_2 \quad (8a)$$

and

$$(c_1 - c_2)^2 + 4\epsilon c_1 c_2 > 0 \quad (8b)$$

CASE II - real roots; $\lambda_1 = \lambda_2 = -M < 0$

In this case fig.(1) can be redrawn as shown in fig.(3).

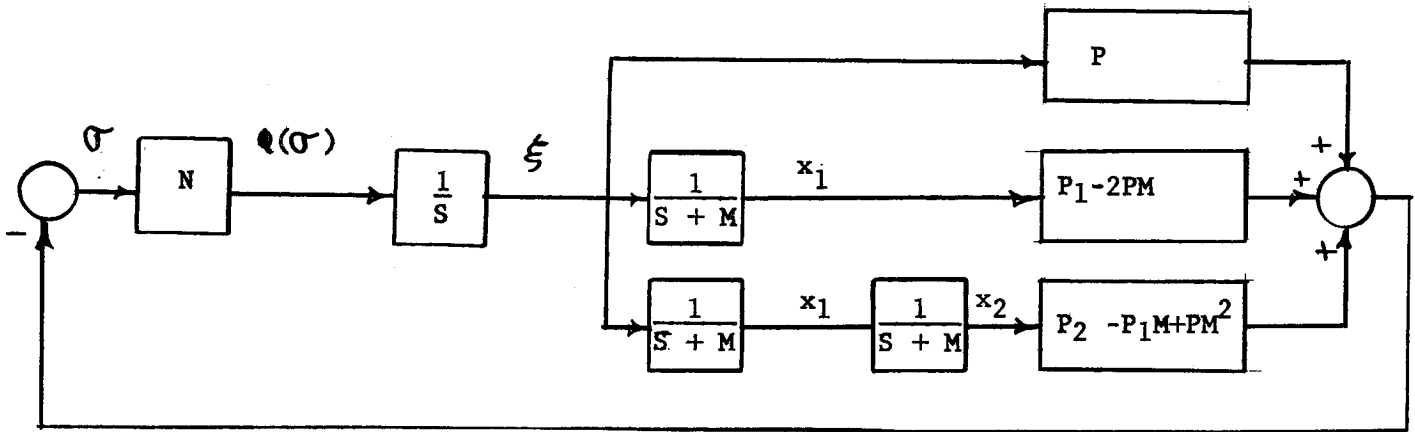


fig. (3)

The equations of motion are

$$\begin{aligned}\dot{x}_1 &= -Mx_1 + \xi \\ \dot{x}_2 &= x_1 - Mx_2 \\ \sigma &= c_1x_1 + c_2x_2 - P\xi\end{aligned}\tag{9}$$

Where

$$\begin{aligned}c_1 &= 2PM - P_1 \\ c_2 &= MP_1 - P_2 - PM^2\end{aligned}$$

Choose

$$C = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$$

For $C > 0$ we require

$$p > 0 \text{ and } pr - q^2 > 0\tag{10}$$

Since $C > 0$, $B > 0$. If

$$B = \begin{pmatrix} p_o & q_o \\ q_o & r_o \end{pmatrix}$$

the relationship between p , q , r and p_o , q_o , r_o is obtained from

$$\begin{pmatrix} p_o & q_o \\ q_o & r_o \end{pmatrix} \begin{pmatrix} -M & 0 \\ 1 & -M \end{pmatrix} + \begin{pmatrix} -M & 1 \\ 0 & -M \end{pmatrix} \begin{pmatrix} p_o & q_o \\ q_o & r_o \end{pmatrix} = - \begin{pmatrix} p & q \\ q & r \end{pmatrix}$$

Which yields

$$2(p_0 M - q_0) = p, (2q_0 M - r_0) = q, 2 r_0 M = r \quad (11)$$

In view of the second inequality of (10) the first can be replaced by $r > 0$. Hence using (11) inequalities (10) can be replaced by

$$4r_0 M(p_0 M - q_0) - (2q_0 M - r_0)^2 > 0, r_0 > 0 \quad (12)$$

Again eq.(4) is taken as the required sufficient condition for absolute stability. Substitution into (4) leads to

$$p_0 = \frac{1}{2} c_1 \quad (13)$$

$$q_0 = \frac{1}{2} c_2$$

Substitution of (13) into (12) yields

$$g(r_0) = r_0^2 - 2 M (c_1 M + c_2) r_0 + (c_2^2 M^2 - 2 c_2 M) < 0 \quad (14a)$$

$$r_0 > 0 \quad (14b)$$

The discriminant of $g(r_0) = 0$ is

$$\delta = 4 c_1^2 M^4 - 8 c_1 c_2 M^3 - 8 c_2^2 M$$

If $\delta \leq 0$, inequality (14a) has no solution. If $\delta > 0$ then $g(r_0)$ has two distinct real roots r_1, r_2 and (14a) can be satisfied by selecting r_0 in the interval (r_1, r_2) . Inequality (14b) must also be satisfied note that $g(0) = c_2^2 M^2 - 2c_2 M$. If $c_2^2 M^2 - 2c_2 M < 0$ then (14b) can clearly be satisfied. On the other hand if $c_2^2 M^2 - 2c_2 M \geq 0$ we must also require $\frac{1}{2} (r_1 + r_2) > 0$ or $M(c_1 M - c_2) > 0$. Hence, we have demonstrated absolute stability if

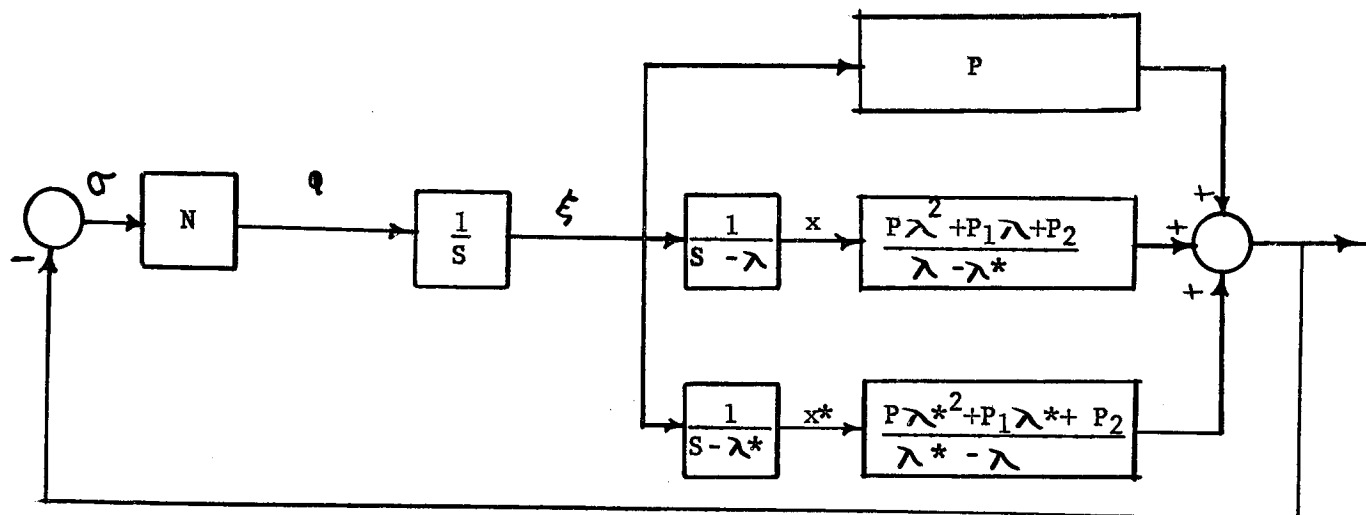
$$c_1^2 M^3 - 2c_1 c_2 M^2 - 2c_2 > 0 \quad (15)$$

and

$$c_2 (c_2 M - 2) < 0 \text{ or } c_1 M - c_2 > 0$$

CASE III. Complex roots; $\lambda, \lambda^*, \lambda = -M + iV, M > 0$.

In this case fig.(1) may be redrawn as in fig.(4).



The equations of motion are

$$\begin{aligned}\dot{x} &= \lambda x + \xi \\ \dot{x}^* &= \lambda^* x^* + \xi \\ \sigma &= c_1^* x + c_1 x^* - p \xi\end{aligned}\tag{16}$$

where

$$c_1^* = \frac{P\lambda^2 + P_1\lambda + P_2}{\lambda^* - \lambda}$$

Choose a positive definite hermitian matrix

$$C = \begin{pmatrix} p & q \\ q^* & r \end{pmatrix}, \quad p, r \text{ real}$$

For $C > 0$ we require

$$p > 0, \quad pr - qq^* > 0\tag{17}$$

If

$$B = \frac{1}{2} \begin{pmatrix} p_0 & q_0 \\ q_0^* & r_0 \end{pmatrix}$$

we have

$$p_0 = \frac{P}{M}, \quad q_0 = \frac{q}{M+iv}, \quad r_0 = \frac{r}{M}$$

Eq. (4) yields

$$p_0 + q_0 + c_1 = 0$$

$$r_0 + q_0^* + c_1^* = 0$$

Note that these equations imply $r_0 = p_0^* = p_0$. Also using the first equation we have

$$p = Mp_0 = -M(c_1 + q_0) \quad (18)$$

Inequalities (17) can then be rewritten in terms of q_0

$$\left. \begin{aligned} M^2 (c_1 + q_0)^2 - (M^2 + V^2) q_0 q_0^* &> 0 \\ c_1 + q_0 &< 0 \end{aligned} \right\} \quad (19)$$

Noting that $c_1 + q_0$ must be real we can write

$$c_1 = \alpha + iB, \quad q_0 = \gamma - iB$$

Also, setting $\varepsilon = V^2/M^2$, (19) can be written

$$h(\alpha) = \alpha^2 + 2\gamma\alpha - (1+\varepsilon)B^2 - \varepsilon\gamma^2 > 0 \quad (20a)$$

$$\alpha + \gamma < 0 \quad (20b)$$

Let α_1 and α_2 be the roots of $h(\alpha)$. Since $\alpha_1\alpha_2 < 0$ the roots are real and of opposite sign. Take $\alpha_1 < 0$, $\alpha_2 > 0$ then to satisfy (20a) we must have

$\alpha < \alpha_1$ or $\alpha > \alpha_2$. Now $h(-\gamma) < 0$. Hence $-\gamma$ is between the roots. Then

to satisfy (20b) we must choose γ such that

$$\alpha < \alpha_1 \quad (21)$$

If this is done then this procedure guarantees absolute stability.

Clearly, it is not always possible to satisfy (21). In fact it is readily found that maximum α_1 occurs when $\delta = -\frac{M}{V} B$ and takes the value

$$\alpha_1 = -\frac{V}{M} B \quad (22)$$

which is less than zero. Consider the case where $p, p_1 = 0$. Then

$$C_1 = \alpha + iB = i \frac{P_2}{2V}$$

so that $\alpha = 0$ and (21) cannot be satisfied in view of (22). Suppose only $p = 0$, then

$$C_1 = \alpha + iB = -\frac{P_1}{2} - i \frac{(P_2 - P_1 M)}{2V}$$

Substituting into (21) and (22) we require

$$-\frac{P_1}{2} < \frac{P_2 - P_1 M}{2M}$$

or simply $P_2 > 0$.

Example: 'The Second Bulgakov Problem', References [5, Chapt. 2, par. 5],
[2, chapt. 5, par. 4]

We consider here a system described by the equations

$$\left. \begin{aligned} T^2 \ddot{\eta} + U \dot{\eta} + K \eta &= T^2 \xi \\ \dot{\xi} &= q(\sigma) \\ \sigma &= a\eta + E\dot{\eta} + G^2 \ddot{\eta} - \frac{1}{Q} \xi \end{aligned} \right\} \quad (1)$$

where T^2 characterizes the inertia of the regulated object and a, E, G^2 , and Q are constants of the regulator and U and K are positive constants.

Now, $\ddot{\eta}$ is eliminated from the last equation using the first

$$= (c - \frac{KG^2}{T^2}) \eta + (E - \frac{GU}{T^2}) \dot{\eta} - (\frac{1}{Q} - G^2) \xi \quad (2)$$

Defining the phase $x_1 = \eta$, $x_2 = \dot{\eta}$ we obtain using (1) and (2)

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{K}{T^2} x_1 - \frac{U}{T^2} x_2 + \xi \\ \dot{\xi} &= Q(\sigma) \\ \sigma &= \left(a - \frac{KG^2}{T^2}\right) x_1 + \left(E - \frac{G^2 U}{T^2}\right) x_2 - \left(\frac{1}{\ell} - G^2\right) \xi \end{aligned} \right\} (3)$$

or

$$\begin{aligned} \dot{\bar{x}} &= \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \bar{x} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xi = A \bar{x} - B \xi \\ \dot{\xi} &= Q(\sigma) \end{aligned} \quad (4)$$

$$\sigma = \gamma_1 x_1 + \gamma_2 x_2 - \rho \xi = \frac{1}{c} \bar{x} - \rho \xi$$

where

$$\alpha_1 = \frac{K}{T^2}, \quad \alpha_2 = \frac{U}{T^2}$$

$$\gamma_1 = a - \frac{KG^2}{T^2}, \quad \gamma_2 = E - \frac{G^2 U}{T^2}, \quad \rho = \frac{1}{\ell} - G^2$$

Noting that α_1 and α_2 are positive it is seen that the roots of the characteristic equation

$$\lambda^2 + \alpha_1 \lambda + \alpha_2 = 0$$

have negative real parts and hence A is stable. Let B and C be given by

$$B = \begin{pmatrix} p_0 & q_0 \\ q_0 & r_0 \end{pmatrix}, \quad C = \begin{pmatrix} p & q \\ q & r \end{pmatrix}$$

We know that $B > 0$ if $C > 0$ and the conditions that $C > 0$ are

$$\left. \begin{aligned} p &> 0 \\ pr - q^2 &> 0 \end{aligned} \right\} \quad (5)$$

The relationship between the elements of B and C is given by Liapunov's matrix equation

$$A^t B + B A = -C$$

$$\begin{pmatrix} -2\alpha_2 q_0 & p_0 - \alpha_1 q_0 - \alpha_2 r_0 \\ p_0 - \alpha_1 q_0 - \alpha_2 r_0 & 2(q_0 - \alpha_1 r_0) \end{pmatrix} = - \begin{pmatrix} p & q \\ q & r \end{pmatrix}$$

or

$$p = 2\alpha_2 q_0$$

$$q = \alpha_1 q_0 + \alpha_2 r_0 - p_0 \quad (6)$$

$$r = 2(\alpha_1 - r_0 - q_0)$$

We have absolute stability if

$$\rho > \bar{d}^t c^{-1} \bar{d}, \quad \bar{d} = B\bar{B} - \frac{1}{2} \bar{c} \quad (7)$$

Choosing as sufficient conditions for absolute stability $\bar{d} = 0$ and $\rho > 0$ we have

$$\bar{d} = - \begin{pmatrix} q_0 + \frac{1}{2} \delta_1 \\ r_0 + \frac{1}{2} \delta_2 \end{pmatrix} = 0$$

and from (6)

$$q_0 = \frac{p}{2\alpha_2}, \quad r_0 = \frac{r}{2\alpha_1} + \frac{p}{2\alpha_1\alpha_2}$$

so that we require

$$\begin{aligned} p + \alpha_2 \delta_1 &= 0 \\ p + r\alpha_2 + \alpha_1\alpha_2 \delta_2 &= 0 \end{aligned} \quad (8)$$

Also, from the $\rho > 0$ condition we require

$$\frac{1}{\rho} > G^2 \quad (9)$$

Noting the first relation of (5) the first equation in (8) yields

$$\delta_1 < 0 \quad (10)$$

Eliminating p between the two equations in (8)

$$r = \delta_1 - \alpha_1 \delta_2$$

and noting that the second equation in (5) requires $r > 0$ we have

$$\delta_1 > \alpha_1 \delta_2 \text{ (note this implies } \delta_2 < 0 \text{)} \quad (11)$$

In terms of the initial constants the sufficient conditions for absolute stability ((9), (10) and (11)) are

$$\begin{aligned} \frac{1}{\ell} &> G^2 \\ E &< \frac{G^2 U}{T^2} \end{aligned} \quad (12)$$

$$0 < \frac{K G^2}{T^2} - \alpha < \frac{U}{T^2} \left(\frac{G^2 U}{T^2} - E \right)$$

Example; Gibson and Rekasius [6]

Consider the closed loop system shown in fig.(1) where the nonlinear element is a saturating amplifier

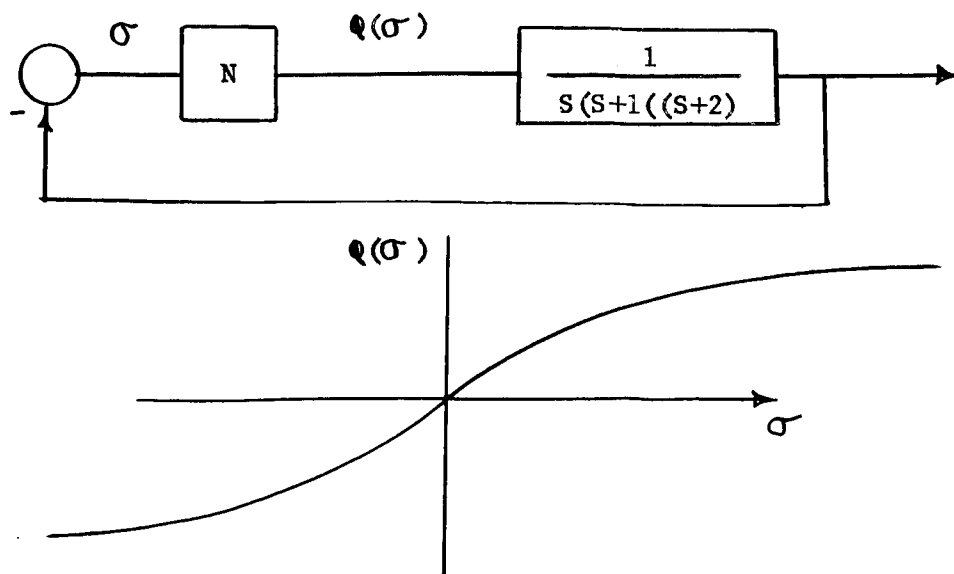


fig. (1)

This system can be treated as an indirect control system and can be put in canonical form by redrawing as in fig. (2).

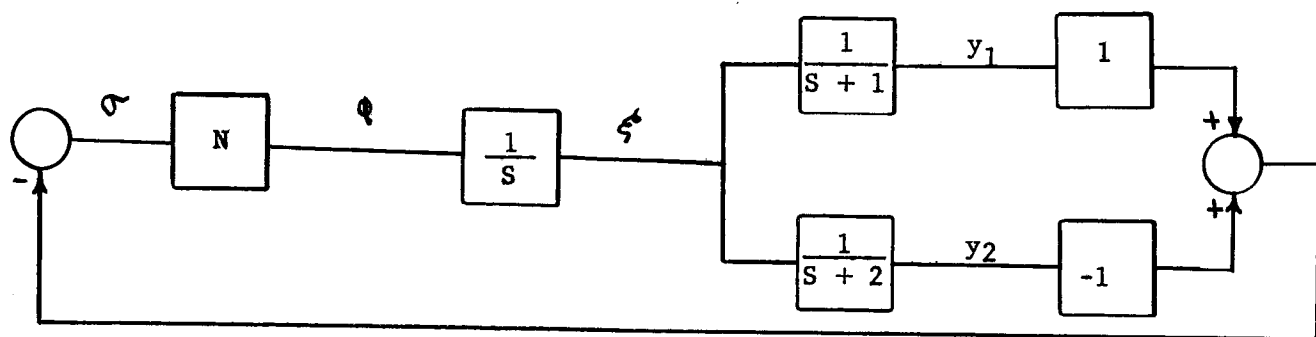


fig. (2)

We have then

$$\begin{aligned} \dot{\bar{y}} &= \begin{pmatrix} -0 & 0 \\ 0 & -2 \end{pmatrix} \bar{y} - \begin{pmatrix} -1 \\ -1 \end{pmatrix} \xi \\ \dot{\xi} &= Q(\sigma) \\ \sigma &= (-1, 1) \bar{y} \end{aligned}$$

We would like to use Lur'e's theorem to demonstrate absolute stability. In this case the 'limit' equations are

$$\begin{aligned} -\frac{a_1^2}{2} - \frac{1}{3} a_1 a_2 + \frac{1}{2} &= 0 \\ -\frac{1}{3} a_1 a_2 - \frac{1}{4} a_2^2 - \frac{1}{2} &= 0 \end{aligned}$$

Since the λ_i are real we look for a real solution to these eqs. Solving simultaneously we find

$$\begin{aligned} a_1 &= -\sqrt{2} - j \\ a_2 &= -\sqrt{2} + j2 \end{aligned}$$

Hence we have failed to show absolute stability. Since Lur'e's theorem gives only sufficient conditions we cannot conclude that the system is not absolutely stable. However, note that in order to be classified as absolutely stable the system must

by globally asymptotically stable for all admissible characteristics Q and a function Q is admissible if it satisfies the three conditions

$$\begin{aligned} Q(0) &= 0 \\ \sigma Q(\sigma) &> 0 \text{ for } \sigma \neq 0 \\ \int_0^{+\infty} Q(\sigma) d\sigma &\text{ diverges} \end{aligned}$$

Hence the linear characteristic $Q = K\sigma$ for $0 < K < \infty$ is admissible. We know that with the linear characteristic, for sufficiently large K the system becomes unstable. Hence, we will not be able to demonstrate absolute stability by any means.

5. Restriction of $Q(\sigma)$ to Finiteness of the Ratio $Q(\sigma)/\sigma$

It has been seen that the usefulness of the presented sufficient conditions for absolute stability are severely limited by the fact that many control systems are only conditionally stable, i.e., with $Q = K\sigma$ they are stable only within some range of K . This region of K is readily found by techniques of linear analysis. Since $Q(\sigma) = K\sigma$ is an admissible characteristic for $0 < K < +\infty$ in our present formulation, we require that the system be asymptotically stable for all $0 < K < +\infty$ in order to be classified as absolutely stable. Hence we obviously rule out a large class of systems of interest.

We attempt to correct this difficulty to some extent by restricting the class of admissible characteristics $Q(\sigma)$ to those contained completely in the sector of the Q - σ plane between the lines $Q = 0$ and $Q = K\sigma$. In other words the condition $\sigma Q(\sigma) > 0$, $\sigma \neq 0$ is replaced by $0 < \frac{Q(\sigma)}{\sigma} < K$. Our attention is being directed toward systems which are stable for linear characteristics with slope less than K . The value of K can be determined by linear methods.

The approach taken is to select V positive definite for all \bar{x} and σ (x in the direct control case) as before but to require $-\dot{V}$ ($S(x, Q)$ in the direct control case) to be positive definite only for a function Q contained in the sector $[0, K]$. We consider the direct and indirect control cases separately.

Indirect Control - The system of equations we have been using to describe an indirect control system is

$$\left. \begin{aligned} \dot{\bar{y}} &= A\bar{y} - \bar{b}\xi \\ \dot{\xi} &= Q(\sigma) \\ \sigma &= \bar{c}_0^t \bar{y} - \rho\xi \end{aligned} \right\} \quad (1)$$

It is more convenient here to work with the eqs. in a somewhat different form.

We define the new variables \bar{x}

$$\bar{x} = \dot{\bar{y}} = A\bar{y} - \bar{b} \quad (2)$$

Then (1) becomes

$$\left. \begin{aligned} \dot{\bar{x}} &= A\bar{x} - \bar{b}Q \\ \dot{\xi} &= Q(\sigma) \end{aligned} \right\} \quad (3)$$

$$\sigma = \bar{c}_0^t A^{-1} \bar{x} + \bar{c}_0^t A^{-1} \bar{b} \xi - \rho \xi = \bar{c}^t \bar{x} - \gamma \xi$$

$$\text{where } \bar{c}^t = \bar{c}_0^t A^{-1}, \quad \gamma = \rho - \bar{c}_0^t A^{-1} \bar{b}$$

We choose a slightly different form for the Liapunov function

$$V(\bar{x}, \xi) = \bar{x}^t B \bar{x} + \alpha (\sigma - \bar{c}^t \bar{x}) (\sigma - \bar{x}^t \bar{c}) + \beta \Phi(\sigma) \quad (4)$$

where α and β are positive constants. Note that $(\sigma - \bar{c}^t \bar{x}) (\sigma - \bar{x}^t \bar{c}) = |\sigma - \bar{c}^t \bar{x}|^2$.

The modification of V is to enable us to obtain a convenient form for \dot{V} .

Differentiating (4)

$$\dot{V} = \frac{\dot{\sigma}}{\bar{x}}^t B \bar{x} + \frac{\dot{\sigma}}{\bar{x}}^t B \frac{\dot{\sigma}}{\bar{x}} - \alpha \delta Q(\sigma - \bar{x}^t \bar{c}) - \alpha \delta Q(\sigma - \bar{c}^t \bar{x})$$

$$+ \beta Q(\bar{c}^t A \bar{x} - \bar{c}^t \bar{b} Q - \delta Q)$$

$$= \frac{\dot{\sigma}}{\bar{x}}^t A B \bar{x} - \frac{\dot{\sigma}}{\bar{b}}^t B \bar{x} Q + \frac{\dot{\sigma}}{\bar{x}}^t B A \bar{x} - \frac{\dot{\sigma}}{\bar{x}}^t B \bar{b} Q$$

$$- \alpha \delta Q(\sigma - \bar{x}^t \bar{c}) - \alpha \delta Q(\sigma - \bar{c}^t \bar{x})$$

$$+ \beta Q(\bar{c}^t A \bar{x} - \rho Q)$$

$$-\dot{V} = \frac{\dot{\sigma}}{\bar{x}}^t C \bar{x} + (\bar{b}^t B - \alpha \delta \bar{c}^t - B \bar{c}^t A) \bar{x} Q + Q \frac{\dot{\sigma}}{\bar{x}}^t (B \bar{b} - \alpha \delta \bar{c})$$

$$+ 2 \alpha \delta Q \sigma + B \rho Q^2$$

$$= \frac{\dot{\sigma}}{\bar{x}}^t C \bar{x} + (B \bar{b} - \alpha \delta \bar{c} - \frac{1}{2} B A^t \bar{c})^t \bar{x} Q$$

$$+ Q \frac{\dot{\sigma}}{\bar{x}}^t (B \bar{b} - \alpha \delta \bar{c} - \frac{1}{2} B A^t \bar{c})$$

$$+ 2 \alpha \delta Q + B \rho Q^2$$

where $-C = A^t B + B A$

and we have used the fact that $\bar{c}^t A \bar{x} = \frac{\dot{\sigma}}{\bar{x}}^t A^t \bar{c}$. Now, add and subtract the quantity $2 \alpha \delta (\sigma - \frac{Q(\sigma)}{K}) Q(\sigma)$

$$-\dot{V} = \left[\frac{\dot{\sigma}}{\bar{x}}^t C \bar{x} + \bar{d}_0^t \bar{x} Q + Q \frac{\dot{\sigma}}{\bar{x}}^t \bar{d}_0 + \left(\beta \rho + \frac{2 \alpha \delta}{K} \right) Q^2 \right]$$

$$\text{where } \bar{d}_0 = B \bar{b} - \alpha \delta \bar{c} - \frac{1}{2} A^t \bar{c} + 2 \alpha \delta \left(\sigma - \frac{Q(\sigma)}{K} \right) Q(\sigma) \quad (5)$$

We consider the case where $\delta > 0$, then since the last term is positive if and only if $Q < K\sigma$ we have a $-\dot{V}$ which is positive definite in the desired sector of the $Q - \sigma$ plane if the term in brackets is positive definite.

first approach - Eq.(5) can be rewritten

$$-V = \begin{pmatrix} \bar{x}^t & q \end{pmatrix} \begin{pmatrix} C & \bar{d}_0 \\ \bar{d}_0^t & \beta\rho + \frac{2\alpha\delta}{K} \end{pmatrix} \begin{pmatrix} \bar{x} \\ q \end{pmatrix} + 2\alpha\delta(\sigma - \frac{q(\sigma)}{K}) q(\sigma) \quad (6)$$

From which it is seen that $-V$ is positive definite in the sector $[0, k]$ if

$C > 0$ and the following Lefshetz inequality holds

$$\beta\rho + \frac{2\alpha\delta}{K} > \bar{d}_0^t C^{-1} \bar{d}_0 \quad (7)$$

Hence these are sufficient conditions for absolute stability.

second approach - This is essentially a generalization of Lur'e's theorem. We wish to make

$$P(\bar{x}, \sigma) = \bar{x}^t C \bar{x} + \bar{d}_0^t \bar{x} q + q \bar{x}^t \bar{d}_0 + (\beta\rho + \frac{2\alpha\delta}{K}) q^2$$

positive definite with respect to (\bar{x}, σ) . Take

$$C = \bar{a} \bar{a}^t + \text{diag}(A_1, \dots, A_n) \quad (8)$$

then

$$P(\bar{x}, \sigma) = \bar{x}^t \text{diag}(A_1, \dots, A_n) \bar{x} + \bar{x}^t \bar{a} \bar{a}^t \bar{x} + q \bar{d}_0^t \bar{x} + \bar{x}^t \bar{d}_0 q + (\beta\rho + \frac{2\alpha\delta}{K}) q^2$$

Add and subtract $\sqrt{\beta\rho + \frac{2\alpha\delta}{K}} (\bar{x}^t \bar{a} + \bar{a}^t \bar{x})$ to $P(\bar{x}, \sigma)$

$$\begin{aligned} P(\bar{x}, \sigma) &= \bar{x}^t \text{diag}(A_1, \dots, A_n) \bar{x} + \left| \bar{a}^t \bar{x} + \sqrt{\beta\rho + \frac{2\alpha\delta}{K}} \right|^2 \\ &+ (q \bar{d}_0^t \bar{x} + \bar{x}^t \bar{d}_0 q) - \sqrt{\beta\rho + \frac{2\alpha\delta}{K}} (\bar{x}^t \bar{a} + \bar{a}^t \bar{x}) \\ &= \bar{x}^t \text{diag}(A_1, \dots, A_n) \bar{x} + \left| \bar{a}^t \bar{x} + \sqrt{\beta\rho + \frac{2\alpha\delta}{K}} \right|^2 \\ &+ q (\bar{d}_0 - \sqrt{\beta\rho + \frac{2\alpha\delta}{K}} \bar{a})^t \bar{x} + q \bar{x}^t (\bar{d}_0 - \sqrt{\beta\rho + \frac{2\alpha\delta}{K}} \bar{a}) \end{aligned}$$

which is positive definite if $A_i > 0$ and

$$\bar{\alpha}_0 - \sqrt{\beta \rho + \frac{2\alpha\gamma}{K}} \bar{\alpha} = 0 \quad (9)$$

or

$$B \bar{b} - \alpha\gamma \bar{c} - \frac{1}{2} A^t \bar{c} - \sqrt{\beta \rho + \frac{2\alpha\gamma}{K}} \bar{\alpha} = 0 \quad (10)$$

Hence with $A_i > 0$ if we can select a set of $\bar{\alpha}_i$ $i = 1, \dots, n$ which satisfy (10) the system is absolutely stable in the sector $[0, k]$.

Direct Control - Consider a direct control system described by the equations

$$\dot{\bar{x}} = A \bar{x} - \bar{b} Q(\sigma) \quad (11)$$

$$\sigma = \frac{1}{c} \bar{x} \Rightarrow \dot{\sigma} = \frac{1}{c} A \bar{x} - \frac{1}{c} \bar{b} Q(\sigma)$$

Take as the Liapunov function

$$V(\bar{x}) = \frac{1}{2} \bar{x}^t B \bar{x} + \beta \Phi(\sigma)$$

Then

$$\begin{aligned} \dot{V} &= \frac{1}{2} \bar{x}^t C \bar{x} + \bar{b}^t B \bar{x} Q + \frac{1}{2} \bar{x}^t B \bar{b} Q \\ &\quad - \beta Q \left(\frac{1}{c} A \bar{x} - \frac{1}{c} \bar{b} Q \right) \\ &= \frac{1}{2} \bar{x}^t C \bar{x} + \left(\bar{b}^t B - \frac{1}{2} \beta \frac{1}{c} A^t \bar{b} \right) \bar{x} Q + Q \frac{1}{2} \bar{x}^t \left(B \bar{b} - \frac{1}{2} \beta A^t \bar{c} \right) \\ &\quad + \beta \frac{1}{c} \bar{b}^t \bar{b} Q^2 \\ &= \frac{1}{2} \bar{x}^t C \bar{x} + \bar{\alpha}^t \bar{x} Q + Q \frac{1}{2} \bar{x}^t \bar{\alpha} + \beta \frac{1}{c} \bar{b}^t \bar{b} Q^2 \end{aligned} \quad (12)$$

where $\bar{\alpha} = B \bar{b} - \frac{1}{2} \beta A^t \bar{c}$, $-C = A^t B + B A$

Add and subtract the quantity $(\sigma - \frac{Q(\sigma)}{K}) Q(\sigma)$ to \dot{V}

$$\begin{aligned} \dot{V} &= \frac{1}{2} \bar{x}^t C \bar{x} + \left[Q \left(\bar{\alpha}^t - \frac{1}{2} \frac{1}{c} A^t \bar{c} \right) \bar{x} + \frac{1}{2} \bar{x}^t \left(\bar{\alpha} - \frac{1}{2} \bar{c} \right) Q \right] \\ &\quad + \left(\beta \frac{1}{c} \bar{b}^t \bar{b} + \frac{1}{K} \right) Q^2 + \left(\sigma - \frac{Q}{K} \right) Q \end{aligned}$$

Then $-\dot{V}$ is positive definite in the sector $[0, K]$ if we make the quantity

$$s(\bar{x}, Q) = \bar{x}^t C \bar{x} + \left[Q(\bar{d} - \frac{1}{2} \bar{c})^t \bar{x} + \bar{x}^t (\bar{d} - \frac{1}{2} \bar{c}) Q \right] + (\beta \bar{c}^t \bar{b} + \frac{1}{K}) Q^2 \quad (13)$$

positive definite in (\bar{x}, Q) .

Consider the case $r = \beta \bar{c}^t \bar{b} + \frac{1}{K} > 0$

first approach - rewrite (13) in the form

$$s(\bar{x}, Q) = (\bar{x}, Q) \begin{pmatrix} C & \bar{d} - \frac{1}{2} \bar{c} \\ (\bar{d} - \frac{1}{2} \bar{c})^t & (\beta \bar{c}^t \bar{b} + \frac{1}{K}) \end{pmatrix} \begin{pmatrix} \bar{x} \\ Q \end{pmatrix} \quad (14)$$

which is positive definite if $C > 0$ and the following Lefschetz inequality holds

$$\beta \bar{c}^t \bar{b} + \frac{1}{K} > (\bar{d} - \frac{1}{2} \bar{c})^t C^{-1} (\bar{d} - \frac{1}{2} \bar{c}) \quad (15)$$

Note the simplest way to insure (15) is to require

$$\bar{d} - \frac{1}{2} \bar{c} = 0$$

second approach - add and subtract the quantity

$$\frac{1}{r} \bar{x}^t (\bar{d} - \frac{1}{2} \bar{c}) (\bar{d} - \frac{1}{2} \bar{c})^t \bar{x}$$

to $S(\bar{x}, Q)$. Then

$$s(\bar{x}, Q) = \left| \sqrt{r} Q + \frac{1}{\sqrt{r}} (\bar{d} - \frac{1}{2} \bar{c})^t \bar{x} \right|^2 + \bar{x}^t C \bar{x} - \frac{1}{r} \bar{x}^t (\bar{d} - \frac{1}{2} \bar{c}) (\bar{d} - \frac{1}{2} \bar{c})^t \bar{x}$$

To make $S(\bar{x}, Q)$ positive definite we need make

$$Q(\bar{x}) = \bar{x}^t C \bar{x} - \frac{1}{r} \bar{x}^t (\bar{d} - \frac{1}{2} \bar{c}) (\bar{d} - \frac{1}{2} \bar{c})^t \bar{x} \quad (16)$$

positive definite.

Take $\bar{c} = \bar{a} \bar{a}^t + \text{diag} (A_1, \dots, A_n)$

then $Q(\bar{x})$ can be written

$$Q(\bar{x}) = \bar{x}^t \text{diag} (A_1, \dots, A_n) + \bar{x}^t \left[\bar{a} \bar{a}^t - \frac{1}{r} (\bar{d} - \frac{1}{2} \bar{c}) (\bar{d} - \frac{1}{2} \bar{c})^t \bar{x} \right]$$

with $A_i > 0$ we insure $Q(\bar{x})$ positive definite by select the elements of \bar{a} so that

$$\frac{1}{r} (\bar{d} - \frac{1}{2} \bar{c}) = \bar{a} \quad (17)$$

or

$$\sqrt{\frac{1}{K} + \beta \bar{c}^t \bar{b}} \bar{a} = \bar{d} - \frac{1}{2} \bar{c} = B \bar{b} - \frac{1}{2} \beta A^t \bar{c} - \frac{1}{2} c \quad (18)$$

Let the system (11) be in Lur'e's first canonical form so that

$$A = \text{diag} (\lambda_1, \dots, \lambda_n)$$

$$\bar{c} = \begin{pmatrix} -\alpha_1^* \\ \vdots \\ \vdots \\ \vdots \\ -\alpha_n^* \end{pmatrix} \quad \bar{b} = \begin{pmatrix} -1 \\ \vdots \\ \vdots \\ \vdots \\ -1 \end{pmatrix}$$

$$B = - \left(\frac{a_j^* a_k}{\lambda_j^* + \lambda_k} \right) + \text{diag} \left(\frac{A_1}{2M_1}, \dots, \frac{A_n}{2M_n} \right) \quad -M_i = \text{Re} \lambda_i < 0$$

The (18) can be written as the n scalar equations

$$a_k \sum_{j=1}^n \frac{a_j^*}{\lambda_j^* + \lambda_k} + \frac{\beta \lambda_k \alpha_k^*}{2} + \frac{\alpha_k^*}{2} - a_k \sqrt{\frac{1}{k} + \beta \bar{c}^t \bar{b}} - \frac{A_k}{2M_k} = 0$$

k = 1, 2, ..., n (19)

which are the 'prelimit' equations.

The 'limit' equations are obtained by setting $A_1 = 0$

$$a_k \sum_{j=1}^n \frac{a_j^*}{\lambda_j^* + \lambda_k} + \frac{\beta \lambda_k \alpha_k^*}{2} + \frac{\alpha_k^*}{2} - \sqrt{\frac{1}{K} + \beta \bar{c}^t b} \bar{a}_k = 0$$

$$K = 1, 2, \dots, n \quad (20)$$

For the case $r = \frac{1}{K} + \beta \bar{c}^t \bar{b} = 0$ we need only set $\bar{d} - 1/2 \bar{c} = 0$ to make $S(\bar{x}, Q)$ positive definite. The resultant equations are the same as (19) and (20) when r is set equal to zero. Hence (19) and (20) hold for both cases.

Example: We return now to the problem considered previously

$$G(S) = \frac{1}{S(S+1)(S+2)}$$

$$\begin{aligned} \dot{\bar{y}} &= \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \bar{y} - \begin{pmatrix} -1 \\ -1 \end{pmatrix} \xi \\ \dot{\xi} &= Q(\sigma) \\ \sigma &= (-1, 1) \bar{y} \end{aligned}$$

It is easily verified that for a linear characteristic the maximum allowable K for stability is $K = 6$. We wish to examine the system for absolute stability in the sector $[0, K]$. Sufficient conditions are that it be possible to select real \bar{a}_i (the realness of \bar{a}_i is imposed since the λ_i in this case are real) which satisfy

$$B \bar{b} - \alpha \gamma \bar{c} - \frac{1}{2} A^t \bar{c} - \sqrt{\beta \rho - \frac{2\alpha \delta^*}{K}} \bar{a} = 0$$

Now

$$\begin{aligned} \bar{c}^t &= \bar{c}_0^t A = (-1, 1) \begin{pmatrix} -1 & 0 \\ 0 & -1/2 \end{pmatrix} = (1, -1/2) \\ \rho &= 0, \gamma = -(1, -1/2) \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 1/2 \end{aligned}$$

Hence we must satisfy the relations

$$-\frac{a_1^2}{2} - \frac{a_1 a_2}{3} - \frac{A_1}{2} - \frac{\alpha}{2} + \frac{1}{2} - \sqrt{\frac{\alpha}{K}} a_1 = 0$$

$$-\frac{a_1 a_2}{3} - \frac{a_2^2}{4} - \frac{A_2}{4} + \frac{\alpha}{4} - \frac{1}{2} - \sqrt{\frac{\alpha}{K}} a_2 = 0$$

or since the only restriction on the A_1 is that that be greater than zero, the a_1 must satisfy the inequalities

$$-\frac{a_1^2}{2} - \frac{a_1 a_2}{3} - \frac{\alpha}{2} + \frac{1}{2} - \sqrt{\frac{\alpha}{K}} a_1 > 0$$

$$-\frac{a_1 a_2}{3} - \frac{a_2^2}{4} + \frac{\alpha}{4} - \frac{1}{2} - \sqrt{\frac{\alpha}{K}} a_2 > 0$$

Select $\alpha = 1$, then

$$-\frac{a_1^2}{2} - \frac{a_1 a_2}{3} - \frac{1}{\sqrt{K}} a_1 > 0$$

$$-\frac{a_1 a_2}{3} - \frac{a_2^2}{4} - \frac{1}{4} - \frac{1}{\sqrt{K}} a_2 > 0$$

or

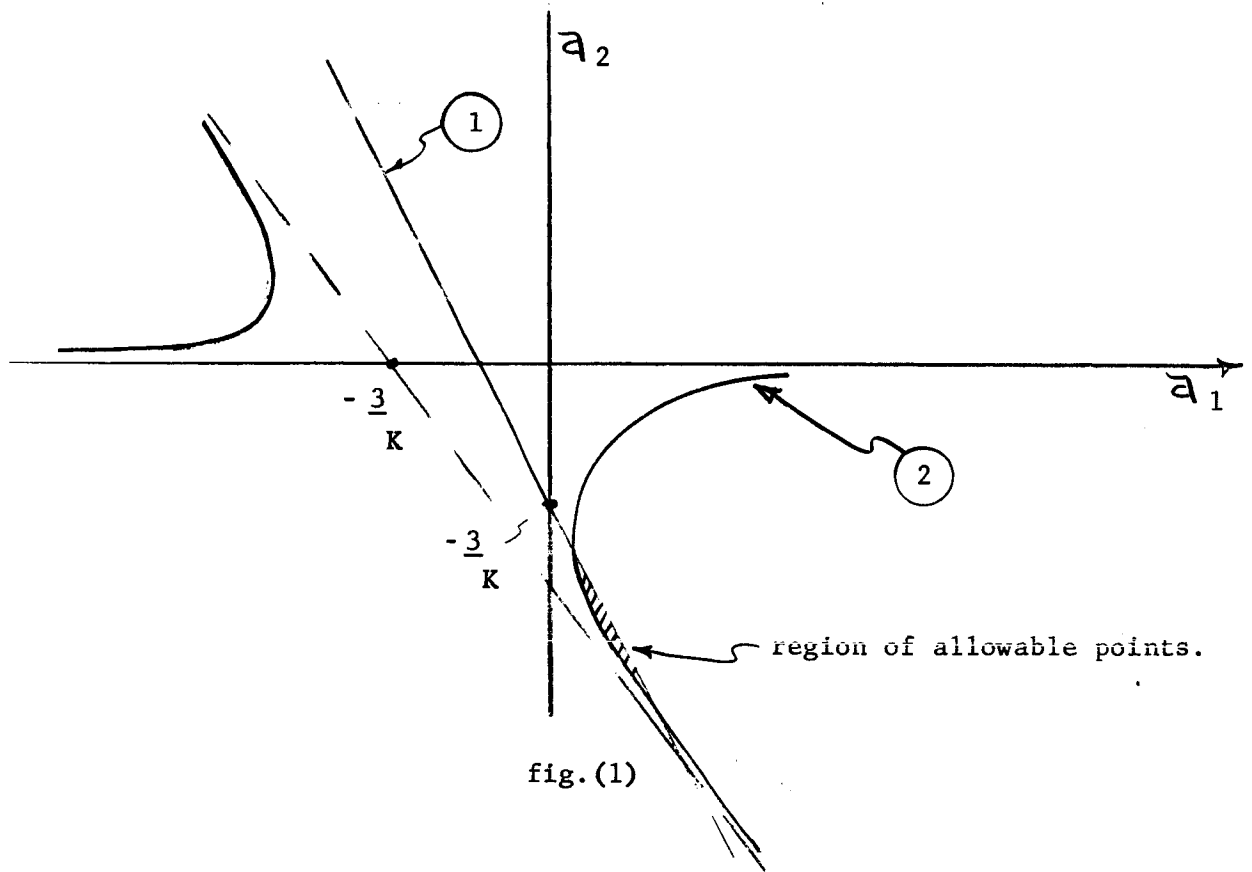
$$a_2 < -\frac{3}{2} a_1 - \frac{3}{\sqrt{K}} \quad \text{for } a_1 > 0$$

$$a_1 < -\frac{3a_2}{4} - \frac{3}{\sqrt{K}} \frac{3}{4a_2} \quad \text{for } a_2 > 0$$

$$a_2 > -\frac{3}{2} a_1 - \frac{3}{\sqrt{K}} \quad \text{for } a_1 < 0$$

$$a_1 > -\frac{3a_2}{4} - \frac{3}{\sqrt{K}} \frac{3}{4a_2} \quad \text{for } a_2 < 0$$

The region of allowable values of (a_1, a_2) is shown in fig. (1)



Clearly the region vanishes as $K \rightarrow \infty$ the limiting case being where (2) is tangent to (1). To find this value of K we solve the relations

$$a_2 = -\frac{3}{2}a_1 - \frac{3}{\sqrt{K}}$$

$$a_1 = -\frac{3}{4}a_2 - \frac{3}{\sqrt{K}} - \frac{3}{4a_2}$$

simultaneously for a_2 which yields

$$a_2 = \frac{-\frac{12}{\sqrt{K}} \pm \sqrt{\frac{144}{K} - 36}}{2}$$

the limiting case occurs when the discriminant vanishes hence

$$K = 4$$

We have demonstrated absolute stability in the sector $[0, 4]$. However we have said nothing about other values of K since the procedure used involves only sufficient conditions. In particular it is possible that the sector could be enlarged by choosing α other than $\alpha = 1$.

6. The Problem of Aizerman

The discussion of the last section raises an interesting question. Suppose the nonlinear characteristic $Q(\sigma)$ is replaced by a linear $Q(\sigma) = K\sigma$ and the system is found, by means of linear analysis, to be stable for $\alpha < K < \beta$. Can we conclude that the system is asymptotically stable in the large for Q contained within the lines $Q = \alpha\sigma$, $Q = \beta\sigma$? In other words, can we conclude that the system is absolutely stable for Q contained in the sector $[\alpha, \beta]$?

This question was originally posed by Aizerman with regard to a specific system of equations. This system being

$$\dot{\bar{x}} = A \bar{x} - \bar{b} Q(x_j) \quad (1)$$

where \bar{b} has only one nonzero element and j is any integer between 1 and n .

Aizerman originally conjectured that the question would have an affirmative answer. However, it has been shown that this is not the case. Aizerman's problem has received much attention, however, the only complete results are available for $n = 2$. In this instance the problem has an affirmative answer except for an exceptional case. For $n \geq 3$ additional conditions must be imposed. Aizerman's problem for $n = 2$ takes the form

$$\dot{x} = f(x) + b y, \dot{y} = c x + d y \quad (2)$$

or

$$\dot{x} = \bar{a} x + f(y), \dot{y} = c x + d y \quad (3)$$

Eqs. (2) were originally treated by Erugin [7] and Malkin [8] and eqs. (3) by Malkin [8]. Malkin's treatment of the problem is repeated in Hahn [9].

Krasovskii [10] considers a more general problem with $n = 2$ and having two nonlinearities. Since this problem has Aizerman's problem as a subclass, Krasovskii's treatment will be given here. A system of two equations with two nonlinear functions will appear in one of the following forms

$$\dot{x} = f_1(x) + b y, \quad \dot{y} = f_2(x) + d y \quad (4)$$

$$\dot{x} = f_1(x) + b y, \quad \dot{y} = c x + f_2(y) \quad (5)$$

$$\dot{x} = f_1(x) + f_2(y), \quad \dot{y} = c x + d y \quad (6)$$

$$\dot{x} = a x + f_1(y), \quad \dot{y} = f_2(x) + d y \quad (7)$$

Case 1. We consider system (4) first. If $f_1(x)$ is replaced by $h_1 x$ and $f_2(x)$ we obtain the corresponding linear system

$$\dot{x} = h_1 x + b y, \quad \dot{y} = h_2 x + d y \quad (8)$$

which is stable provided the Routh-Hurwitz conditions

$$h_1 + d < 0, \quad d h_1 - b h_2 > 0 \quad (9)$$

are satisfied. Hence we postulate the conditions

$$\frac{f_1(x)}{x} + d < 0, \quad d \frac{f_1(x)}{x} - b \frac{f_2(x)}{x} > 0 \quad \text{for } x \neq 0 \quad (10)$$

and inquire into the absolute stability of (4), i.e. are conditions (10) sufficient to guarantee the absolute stability of (4)?

Krasovskii considers the Liapunov function

$$V(x, y) = (d x - b y)^2 + 2 \int_0^x [f_1(\xi) d - b f_2(\xi)] d\xi \quad (11)$$

which is seen to be positive definite, if $b \neq 0$, by virtue of the second inequality of (10). Note that if $b = 0$, the variables of (4) are separated and the system can be integrated directly. This case is not treated here.

Also note that $v(x, y) \rightarrow \infty$ if $x^2 + y^2 \rightarrow \infty$. Differentiating (10) with respect to t and using (4)

$$\begin{aligned}
 \dot{V} &= 2 (d x - b y) (d \dot{x} - b \dot{y}) + 2 (d f_1(x) - b f_2(x)) \dot{x} \\
 &= 2 (d x - b y) (d f_1 - b f_2) + 2 (d f_1 - b f_2) (f_1 + b y) \\
 &= 2 (f_1(x) + d x) (d f_1(x) - b f_2(x)) \\
 &= 2x^2 \left(\frac{f_1(x)}{x} + d \right) \left(d \frac{f_1(x)}{x} - b \frac{f_2(x)}{x} \right)
 \end{aligned} \tag{12}$$

So that $\dot{V}(x, y) < 0$ for $x \neq 0$ and $\dot{V}(x, y) = 0$ for $x = 0$ and hence $\dot{V}(x, y)$ is negative semi-definite. Moreover, since $dx/dt \neq 0$ for $y \neq 0$ the conditions of theorem 5. are satisfied and we have absolute stability.

Note that if only one of the functions $f_1(x)$ or $f_2(x)$ is nonlinear then the problem treated is equivalent to eqs. (2) or (3), i.e., to Aizerman's problem for $n = 2$.

Krasovskii [10] points out that similar methods lead to the following results for eqs. (5), (6) and (7).

Case 2. Consider eqs. (5). The conditions

$$\frac{f_1(x)}{x} + \frac{f_2(y)}{y} < 0, \quad \frac{f_1(x)}{x} \frac{f_2(y)}{y} - b c > 0 \text{ for } x, y \neq 0 \tag{13}$$

are sufficient for absolute stability. Here the Liapunov function is taken as

$$\begin{aligned}
 V(x, y) &= \frac{1}{2} (\beta^2 - b c) x^2 + \frac{1}{2} (b^2 - \frac{b^3 c}{\alpha^2})^2 \\
 &\quad + \beta \int_0^x f_1(\xi) d\xi + \frac{b^2}{\alpha^2} \int_0^y f_2(\eta) d\eta - b \beta x y
 \end{aligned}$$

Case 3. Consider eqs. (6). In this case the stronger form of the conditions corresponding to the Routh-Hurwitz inequalities

$$\frac{f_1(x)}{x} + d \leq -\delta < 0, \quad \frac{f_1(x)}{x} - \frac{f_2(y)}{y} c \geq \delta > 0 \tag{14}$$

for $x, y \neq 0$

are not even sufficient to guarantee asymptotic stability in the small. However,

the additional hypothesis that

$f_1(x) + b x$ is a monotonically decreasing

function of x , or in particular

$$f_1'(x) + b < 0 \quad (15)$$

along with (13) guarantees absolute stability. The proof is based on the Liapunov function

$$V(x, y) = \frac{1}{2} (a x + b y)^2 - a \int_0^y \left[f_1\left(-\frac{b}{a} \xi\right) - f_2(\xi) \right] d\xi$$

Case 4. Consider eqs. (7). Here the conditions

$$a + d < 0, \quad d - \frac{f_1(x)}{x} - \frac{f_2(y)}{y} > 0 \quad \text{for } x, y \neq 0 \quad (16)$$

guarantee absolute stability.

Example 1. [11] Bergen and Williams have verified Aizerman's conjecture for the class of third order control systems illustrated in fig.(1).

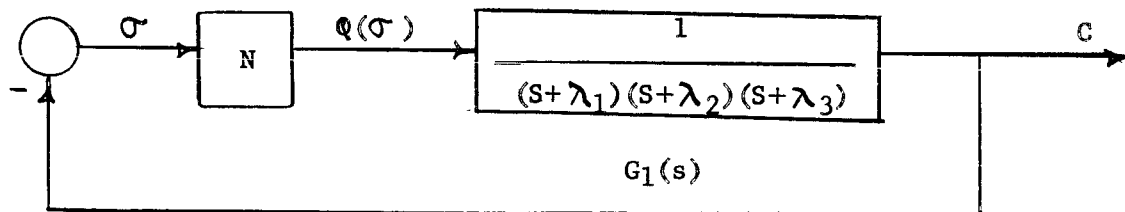


fig.(1)

This system is described by the differential equation

$$\ddot{\sigma} + (\lambda_1 + \lambda_2 + \lambda_3) \dot{\sigma} + (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) \sigma + \lambda_1 \lambda_2 \lambda_3 \sigma + Q(\sigma) = 0 \quad (1)$$

If the nonlinear characteristic $Q(\sigma)$ is replaced by the linear characteristic

$Q = K\sigma$, the linear system is found to be stable for

$$K_1 < K < K_2$$

where

$$K_1 = -\lambda_1 \lambda_2 \lambda_3 \quad (2)$$

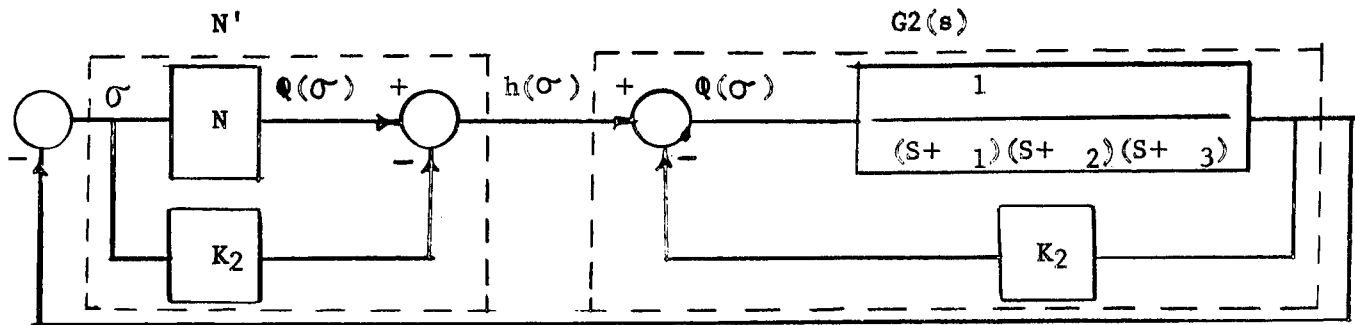
$$K_2 = (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3) - \lambda_1 \lambda_2 \lambda_3$$

We wish to demonstrate the absolute stability of the system under the hypothesis that

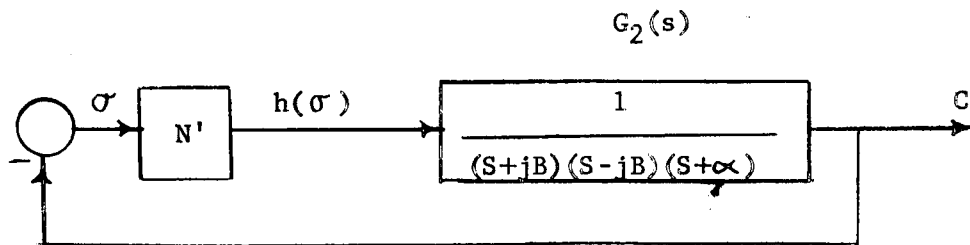
$$K_1 < \frac{Q(\sigma)}{\sigma} < K_2 \quad \text{for } \sigma \neq 0 \quad (3)$$

$$\text{and } Q(0) = 0$$

The system is successively transformed via the diagrams in fig. (2).



(a)



(b)

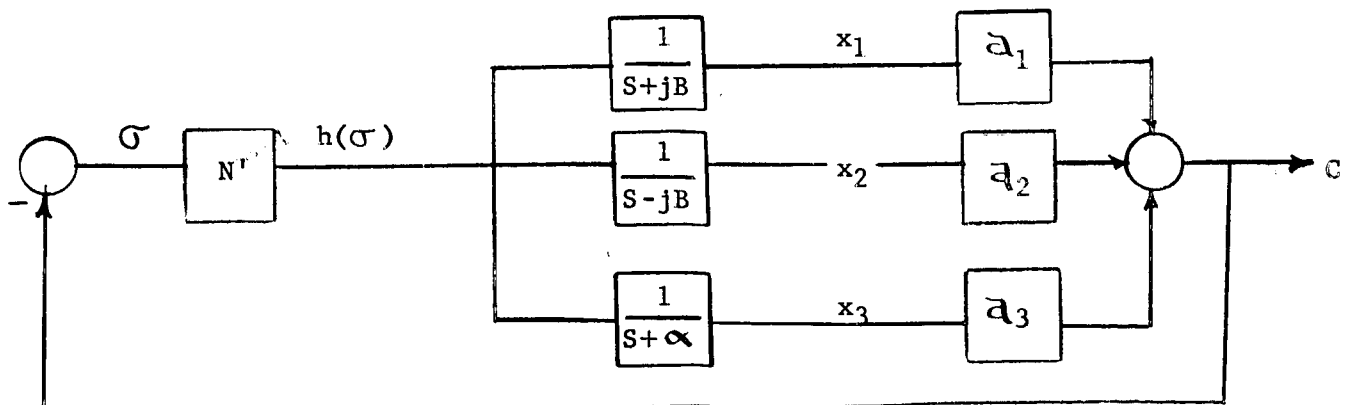


fig. (2) (c)

where

$$h(\sigma) = f(\sigma) - K_2 \sigma$$

$$\alpha = s_1 + s_2 + s_3 \quad (4)$$

$$\beta = \sqrt{s_1 s_2 + s_2 s_3 + s_3 s_1}$$

$$a_1 = \frac{-1}{2j\beta(\alpha - j\beta)}, a_2 = \frac{1}{2j\beta(\alpha + j\beta)}, a_3 = \frac{1}{\alpha^2 + \beta^2} \quad (5)$$

Now eq. (2) can be written as the first order system

$$\dot{\bar{x}} = A \bar{x} + \bar{b} h(\sigma) \quad (6)$$

$$\sigma = -\bar{a}^t \bar{x} = -\bar{x}^t \bar{a}$$

where

$$A = \begin{pmatrix} -j\beta & 0 & 0 \\ 0 & j\beta & 0 \\ 0 & 0 & -\alpha \end{pmatrix}$$

$$\bar{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \bar{a} = \begin{pmatrix} a_1^* \\ a_2^* \\ a_3^* \end{pmatrix}$$

Since Q satisfies (3) we have

$$-\alpha\beta^2 = k_1 - k_2 < \frac{h(\sigma)}{\sigma} < 0 \quad \text{for } \sigma \neq 0 \quad (7)$$

$$h(0) = 0$$

Choose as a candidate for a Liapunov function

$$V(\bar{x}) = \bar{x}^t B \bar{x} + \rho \int_0^\sigma h(N) dN \quad (8)$$

$$\text{with } B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix}$$

where $\rho > 0$ and γ are to be determined. Differentiating (8), using (6) and noting that $\bar{a}^t \bar{b} = 0$ we obtain.

$$\begin{aligned} -\dot{V} = & -\bar{x}^t (A^t \bar{b} + B A) \bar{x} + \frac{1}{2} \left[(-B \bar{b} + \frac{1}{2} \gamma A^t \bar{a})^t \bar{x} \right. \\ & \left. + \bar{x}^t (-B \bar{b} + \frac{1}{2} \gamma A^t \bar{a}) \right] \end{aligned}$$

Now define the function $g(\sigma)$ by

$$\sigma g(\sigma) = \frac{1}{2} \dot{V}$$

so that

$$-\alpha \beta^2 < g(\sigma) < 0 \quad \text{for } \sigma \neq 0$$

and

$$g(0) = 0$$

Then $-\dot{V}$ can be written

$$-\dot{V} = \bar{x}^t \left\{ -(A^t B + B A) + g \left[\begin{pmatrix} \bar{a}^t \bar{b} & B - \frac{1}{2} \gamma \bar{a} \bar{a}^t A \\ (B \bar{b} \bar{a}^t - \frac{1}{2} \gamma A^t \bar{a} \bar{a}^t) \end{pmatrix} \right] \right\} \bar{x} \quad (9)$$

Since $-\dot{V}$ is linear in $g(\sigma)$ we need only insure that $-\dot{V}$ is positive semidefinite at the extreme values of $g(\sigma)$ to be certain that $-\dot{V}$ is positive semidefinite for all intermediate values. Designating the matrix in (9) by Q we have for $g(\sigma) = 0$

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\alpha\rho \end{pmatrix} \quad (10)$$

and for $g(\sigma) = -\alpha\beta^2$

$$Q_1 = \alpha\beta^2 \begin{pmatrix} q_1 & q_2 & q_3 \\ q_2^* & q_1 & q_3^* \\ q_3 & q_3 & q_4 \end{pmatrix} \quad \left. \begin{array}{l} \text{for } \sigma \neq 0 \end{array} \right\} \quad (11)$$

$Q_1 = Q_0 \text{ for } \sigma = 0$

where

$$q_1 = a_3$$

$$q_2 = -a_2 (2 - j \gamma a_2 B)$$

$$q_3 = -[a_3 + \rho a_2 + \frac{\gamma a_2 a_3}{2} (\alpha - j B)]$$

$$q_4 = a_3 \alpha [2 \frac{\alpha}{B^2} \rho - \gamma a_3]$$

Matrix Q_0 is clearly positive semidefinite. Matrix Q_1 is positive semidefinite if the following conditions are satisfied

$$1) q_1 \geq 0$$

$$2) q_4 \geq 0$$

$$3) q_1^2 - q_2^* q_2 \geq 0 \quad (12)$$

$$4) q_1 q_4 - q_3^* q_3 \geq 0$$

$$5) |q_1| \geq 0$$

Condition 1) is satisfied, Condition 3) is satisfied if

$$-\gamma^2 + 8\gamma\alpha - 16\alpha^2 \geq 0 \quad (13)$$

which is satisfied with the equality sign if

$$\gamma = 4\alpha \quad (14)$$

Using (14) condition 4) becomes

$$-\rho^2 + 4\rho - 4 \geq 0 \quad (15)$$

which is satisfied with the equality sign if

$$\rho = 2 \quad (16)$$

Using (14) and (16) conditions (2) and (5) are satisfied. Hence $-V$ is positive semidefinite for

$$-\alpha B^2 \leq g(\sigma) \leq 0 \quad (17)$$

if (14) and (16) hold.

It is readily shown that $V(\bar{x})$ is positive definite. From (4) and (8)

$$\begin{aligned} V(\bar{x}) &\geq \bar{x}^t B \bar{x} - 2 \alpha \beta \sigma^2 = \bar{x}^t \left[B - 2 \alpha \beta \bar{a} \bar{a}^t \right] \bar{x} \\ &= 2 \alpha \beta \frac{1}{2} \bar{x}^t H \bar{x} \end{aligned} \quad (18)$$

where

$$H = \begin{pmatrix} h_1 & h_2 & h_3 \\ h_2^* & h_1 & h_3^* \\ h_3^* & h_3 & h_4 \end{pmatrix} = \frac{1}{2 \alpha \beta} B - \bar{a} \bar{a}^t \quad (19)$$

$$h_1 = \frac{1}{2 \alpha \beta} - a_1 a_2$$

$$h_2 = -a_1^2$$

$$h_3 = -a_1 a_3$$

$$h_4 = \frac{1}{\alpha \beta} - a_3^2$$

H is positive definite if and only if

- 1) $h_1 > 0$
- 2) $h_1^2 - h_2^* h_2 > 0$
- 3) $|H| > 0$

These conditions are satisfied. Note also, that since H is positive definite

$$V(\bar{x}) \longrightarrow \infty \text{ as } \|\bar{x}\| \longrightarrow \infty$$

In order to satisfy the conditions of theorem 5, it must also be shown that the surface

$$F(\bar{x}) = -\dot{V}(\bar{x}) = \bar{x}^t Q \bar{x} = 0 \quad (20)$$

contains no nontrivial half trajectory

$$\bar{x}(\bar{x}_0, t_0, t) \quad 0 < t < \infty$$

of the system.

In order to investigate the nature of the surface (20), $\dot{-V}$ is written in the form

$$\begin{aligned} \dot{-V} &= \bar{x}^t (A^t \bar{b} + B A) \bar{x} + g \left\{ \sigma 2 \operatorname{Re} \left[\left(-B \bar{b} + \frac{1}{2} A^t \bar{a} \right) \bar{x}^t \right] \right\} \\ &= 4\alpha^2 x_3^2 + g \left\{ \sigma 2 \operatorname{Re} \left[(-1-2\alpha a_1 j B) x_1 + (-1 + 2\alpha a_1^* j B) x_2 \right. \right. \\ &\quad \left. \left. + 2(-1 - \alpha^2) a_3 x_3 \right] \right\} \end{aligned}$$

Viewing $\dot{-V}$ as a linear function of g and having shown that $\dot{-V}$ is non-negative at the end points of the permissible domain of g $\dot{-V}$ can vanish in this domain if and only if

$$x_3 = 0 \quad (21)$$

and

$$\sigma = 0 \quad (a) \text{ or } \operatorname{Re} \left[\frac{(jB)}{\alpha - jB} x_1 + \frac{(-jB)}{\alpha + jB} x_2 \right] = 0 \quad (b)$$

where in the last condition we have substituted for a_1 using (5) and have made use of the fact that $x_3 = 0$. Also, since $x_2 = x_1^*$ condition (b) leads to the relation

$$x_1 = \frac{\alpha - jB}{\alpha + jB} x_2 \quad (22)$$

But direct substitution of (21) and (22) into $\sigma = -\bar{a}^t \bar{x}$ shows that σ vanishes and hence condition (a) is a necessary consequence of (b). The converse is also true, that is $\sigma = 0$ insures condition (b). Hence the two are equivalent and we can say that $\dot{-V} = 0$ if and only if $x_3 = 0$ and $\sigma = 0$. Thus if a solution is to remain on this surface it must satisfy (from (6))

$$\dot{x}_1 = -iBx_1, \quad \dot{x}_2 = iBx_2, \quad a_1 x_1 + a_1^* x_2 = 0$$

i.e., there must be a constant such that

$$a_1 \delta e^{-iBt} + a_1^* \delta^* e^{iBt} = 0$$

Since the exponential are linearly independent and $a_1 \neq 0$, we require $\delta = 0$.

Hence all of the required conditions are satisfied and the system of fig.(1) is absolutely stable for admissible Q .

7. Some Theorems of Popov

Indirect Control - Popov [12] considers a system in the form

$$\left. \begin{aligned} \dot{\bar{x}} &= A \bar{x} - b Q(\sigma) \\ \dot{\xi} &= Q(\sigma) \\ \sigma &= \bar{c}^t \bar{x} - \gamma \xi \end{aligned} \right\} \quad (1)$$

This representation of an indirect control system is equivalent to those we have used previously. To see this, the last equation of (1) is differentiated and we obtain

$$\left. \begin{aligned} \dot{\bar{x}} &= A \bar{x} - b Q(\sigma) \\ \dot{\sigma} &= \bar{c}^t A \bar{x} - \rho Q(\sigma) \\ \rho &= \gamma + \bar{c}^t b \end{aligned} \right\} \quad (2)$$

Let $A_s = sI - A$ so that $|A_s| = 0$ is the characteristic equation of A . Also, let A be stable, i.e., its eigenvalues have negative real parts. Then $|A_{iw}| \neq 0$ for all w and hence A_{iw}^{-1} exists.

The scalar function $v(t)$ is defined by

$$v(t) = \bar{c}^t e^{At} \bar{b} \quad (3)$$

The Fourier transform of $v(t)$ is

$$\begin{aligned} N(jw) &= \int_0^{\infty} v(t) e^{-iwt} dt \\ &= \bar{c}^t \left[\int_0^{\infty} e^{-A_{iw}t} dt \right] \bar{b} = -\bar{c}^t A_{iw}^{-1} \left[e^{-A_{iw}t} \right]_0^{\infty} \bar{b} \end{aligned}$$

Now, there is a similarity transformation such that $A = S A_0 S^{-1}$ where A_0 is a Jordan normal form.

Then

$$\left[e^{-A_0 t} \right]_0^\infty = \left[s e^{A_0 t} s^{-1} e^{-i\omega t I} \right]_0^\infty = -I$$

So that

$$N(j\omega) = \bar{c}^t A_{i\omega}^{-1} \bar{b} \quad (4)$$

In addition introduces the function

$$G(j\omega) = N(j\omega) + \frac{\gamma}{j\omega} \quad (5)$$

It is readily shown that $G(s)$ is actually the transfer function of the linear part of the system. Taking the Laplace transform of (2)

$$\begin{aligned} s \bar{x}(s) &= A \bar{x}(s) - \bar{b} Q(s) \\ s \sigma(s) &= \bar{c}^t A \bar{x}(s) - \rho Q(s) \end{aligned} \quad (6)$$

from the first equation we have

$$\bar{x}(s) = -A_s^{-1} \bar{b} Q(s)$$

Then from the second equation

$$\begin{aligned} s \sigma(s) &= -\bar{c}^t A A_s^{-1} \bar{b} Q(s) - \rho Q(s) \\ &= -\left[\bar{c}^t (sI - A_s) A_s^{-1} \bar{b} + \rho \right] Q(s) \\ &= -\left[\bar{c}^t s A_s^{-1} \bar{b} - \bar{c}^t \bar{b} + \rho \right] Q(s) \\ \sigma(s) &= -\left[\bar{c}^t A_s^{-1} \bar{b} + \frac{\rho - \bar{c}^t \bar{b}}{s} \right] Q(s) \\ \sigma(s) &= -\left[\bar{c}^t A_s^{-1} \bar{b} + \frac{\gamma}{s} \right] Q(s) = -G(s) Q(s) \end{aligned} \quad (7)$$

From (7) it is seen that $G(s)$ is the transfer function between Q and $-\sigma$ as it was desired to show.

We state the following two theorems of Popov:

First Theorem of Popov - A sufficient condition for the absolute stability of system (1) is that for some nonnegative number q and all real w we have

$$\operatorname{Re} \{ (1 + iwq) G(iw) \} \geq 0 \quad (8)$$

Second Theorem of Popov - If the absolute stability of the system (1) may be determined by means of a Liapunov function $V(\bar{x}, \sigma)$ of the form "quadratic in \bar{x} , plus $\rho \int_0^\sigma Q(\sigma) d\sigma$ " then there exists a $q \geq 0$ such that (8) holds.

Note that the second theorem implies that satisfaction of the first theorem is necessary if the criterion of the previous sections are satisfied and hence the first theorem can be considered a broader sufficient condition than those obtained previously.

There is an interesting geometric interpretation of the criterion of Popov's first theorem. Let

$$u(w) = \operatorname{Re} G(jw) \quad (9)$$

and

$$v(w) = w \operatorname{Im} G(jw)$$

Then the locus of points (u, v) in the plane of (u, v) is called the 'modified frequency response' or 'modified phase-amplitude characteristic' (M.P.A.C.)

We have from the theorem

$$u(w) + q v(w) \geq 0, \quad q \geq 0$$

Hence, if there is a straight line situated either in the first and the third quadrants of the (u, v) plane or it is on the coordinate axis, and in addition it is such that the M.P.A.C. is "on the right" of this straight line, then the origin of the investigated system is absolutely stable.

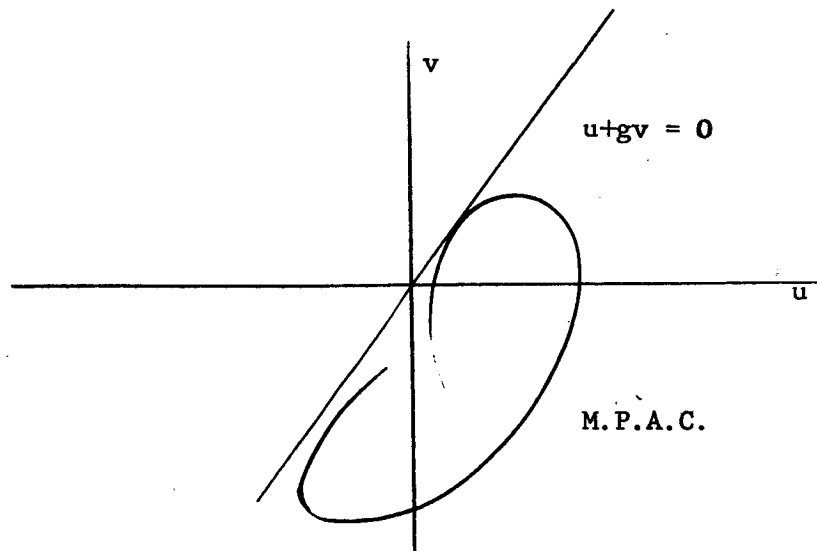


fig. (1)

Proof of the two theorems can be found in [2], [4] or [2].

The Liapunov Function of Popov

The general form of the Liapunov function mentioned in the second theorem is (see [2, p 98] and [4, p. 36]) :

$$V(\bar{x}, \sigma) = \bar{x}^t B \bar{x} + \alpha \sigma^2 + \sigma \bar{f}^t \bar{x} + \beta \Phi(\sigma) \quad (1)$$

which may also be written in the form

$$V(\bar{x}, \sigma) = \bar{x}^t B \bar{x} + \alpha (\sigma - \bar{c}^t \bar{x}) (\sigma - \bar{x}^t \bar{c}) + \beta \Phi'(\sigma) + \sigma \bar{f}^t \bar{x} \quad (2)$$

We will now show that to have $-\dot{V}$ at least positive semidefinite we must have $\bar{f} = 0$. Differentiating (2)

$$\begin{aligned} -\dot{V} = & \bar{x}^t (-A - B - A) \bar{x} + \beta \rho Q + \bar{x}^t (B \bar{b} - \alpha \delta \bar{c} - \frac{1}{2} B A \bar{c}) Q \\ & + Q (B \bar{b} - \alpha \delta \bar{c} - \frac{1}{2} B A \bar{c}) \bar{x} + 2 \alpha \delta \sigma Q \\ & - \frac{d}{dt} (\sigma \bar{f}^t \bar{x}) \end{aligned}$$

$$\begin{aligned} \dot{V} = \bar{x}^t C \bar{x} + \beta \rho Q^2 + \bar{x}^t \bar{d}_0 Q + \bar{d}_0^t \bar{x} Q + 2\alpha \gamma \sigma Q \\ - \frac{d}{dt} (\sigma \bar{f}^t \bar{x}) \end{aligned} \quad (3)$$

where $\bar{d}_0 = B \bar{b} - \left(\frac{1}{2} B A^t \bar{c} + \alpha \gamma \bar{c}\right)$

$$-C = A^t B + B A$$

Also,

$$\frac{d}{dt} (\sigma \bar{f}^t \bar{x}) = (\bar{x}^t A^t \bar{c} - \rho Q) \bar{f}^t \bar{x} + \sigma \bar{f}^t (A \bar{x} - \bar{b} Q)$$

Since \dot{V} is to be at least positive semidefinite in (\bar{x}, σ) we can choose an arbitrary small number ϵ and \dot{V} must be nonnegative when we make the transformation.

$$x \rightarrow \epsilon x, \sigma \rightarrow \sigma, Q \rightarrow \epsilon^2 Q$$

(clearly, if Q is an admissible function $\epsilon^2 Q$ is also admissible). We obtain

$$V = - \bar{f}^t A \bar{x} + O(\epsilon)$$

Now A is nonsingular and hence for $\sigma \neq 0$ and $\bar{f} \neq 0$, ϵ and \bar{x} arbitrary, we have $\epsilon \sigma \bar{f}^t A \bar{x} \neq 0$. Moreover, for small ϵ the sign of \dot{V} will depend on the arbitrary sign of ϵ . Hence we must have $\bar{f} = 0$.

Then finally we have the most general Liapunov function of the form "quadratic in (\bar{x}, σ) plus $\beta \int_0^\sigma Q(\sigma) d\sigma$ "

$$V = \bar{x}^t B \bar{x} + \alpha (\sigma - \bar{c}^t \bar{x})(\sigma - \bar{x}^t \bar{c}) + \beta \Phi(\sigma) \quad (4)$$

$$\dot{V} = \bar{x}^t C \bar{x} + \beta \rho Q^2 + \bar{x}^t \bar{d}_0 Q + \bar{d}_0^t \bar{x} Q + 2\alpha \gamma \sigma Q \quad (5)$$

Furthermore, for V to be positive definite we require besides $B > 0; \alpha \geq 0,$

$$\beta \geq 0, \alpha + \beta > 0$$

Let $\alpha \neq 0$, the substitution $\bar{x} \rightarrow \epsilon \bar{x}, \sigma \rightarrow \sigma, Q \rightarrow \epsilon^2 Q$ yields $V \approx \alpha \sigma^2$ hence $\alpha > 0$

Let $B \neq 0$, the substitution $x \rightarrow \varepsilon^2 x$, $\sigma \rightarrow \varepsilon^2 \sigma$, $Q \rightarrow Q$ yields $V \approx \varepsilon^2 \beta \Phi(\sigma)$, hence $\beta > 0$.

If both $\alpha, \beta = 0$, $V = 0$ for $\bar{x} = 0$ and $\sigma \neq 0$, hence $\alpha + \beta > 0$.

Note we have used the form (4) in Section (5).

9. The Popov Theorems

Direct Control -

In this case we consider a system described by the differential equations

$$\begin{aligned} \dot{\bar{x}} &= A \bar{x} - \bar{b} Q(\sigma) \\ \sigma &= \bar{c}^t \bar{x} \end{aligned} \quad (1)$$

Eqs. (10) are the same as (1) except that $\delta = 0$. We have then

$$G(i\omega) = \frac{1}{c} \begin{bmatrix} t & -1 \\ c & A_{i\omega} \end{bmatrix} \bar{b}$$

and Popov's first theorem holds with this modification of $G(i\omega)$.

The second theorem holds also but with respect to Lur'e-Postnikov Liapunov function

$$V(\bar{x}) = \bar{x}^t B \bar{x} + \beta \Phi(\sigma)$$

constructed as in section 5.

10. Kalman's Theorem

Kalman introduced the following definitions. The system is said to be 'completely controllable' if for fixed A and \bar{b} , $c \bar{A}_s^{-1} \bar{b} \equiv 0$ only if $\bar{c} \equiv 0$. Note that $\bar{c}^t \bar{A}_s^{-1} \bar{b}$ is the transfer function between Q and $\bar{c}^t \bar{x}$. The system is said to be 'completely observable' if for fixed A and \bar{c} , $\bar{c}^t \bar{A}_s^{-1} \bar{b} \equiv 0$ only if $\bar{b} \equiv 0$.

According to Kalman [13] the following statements are equivalent. (a) the pair (A, \bar{b}) is completely controllable, (b) $\det [\bar{b}, A^2 \bar{b}, \dots, A^{n-1} \bar{b}] \neq 0$, (c) $\bar{c}^t e^{At} \bar{b} \equiv 0$ for all t implies $\bar{x} = 0$, (d) \bar{b} does not belong to any proper A -invariant subspace of R^n . He also adopts the definition, (A, \bar{c}) is completely observable if and only if (A^t, \bar{c}) is completely controllable.

In the cited paper Kalman writes Popov's inequality in a modified form in terms of two parameters α, β

$$\operatorname{Re} [(2\alpha\delta + i\omega\beta(G(j\omega))] \geq 0 \text{ for all real } \omega \quad (1)$$

and some pair $\alpha \geq 0, \beta \geq 0, \alpha + \beta > 0$.

He then proves the following

Theorem - Consider the system

$$\dot{\bar{x}} = A \bar{x} - \bar{b} Q(\sigma), \quad \dot{\sigma} = Q(\sigma), \quad \sigma = \bar{c}^t \bar{x} - \delta \int_0^\sigma Q(\sigma) d\sigma \quad (2)$$

where $\delta > 0$, A is stable, (A, \bar{b}) is completely and (A, \bar{c}) is completely observable. We seek a suitable Liapunov function of the Popov type.

$$V(\bar{x}, \sigma) = \bar{x}^t B \bar{x} + \alpha (\sigma - \bar{c}^t \bar{x})^2 + \beta \int_0^\sigma Q(\sigma) d\sigma \quad (3)$$

- (A) $V > 0$ and $\dot{V} \leq 0$ for any admissible Q (hence V is a Liapunov function which assures Liapunov stability of $\bar{x} = 0$ of (2) for any admissible Q) if and only if (1) holds.
- (B) Suppose V satisfies the preceding conditions. Then V is a Liapunov function which assures absolute stability of (1) if and only if either a) $\alpha \neq 0$ or b) $\alpha = 0$ and the equality sign in (1) occurs only at these values of ω where

$$\operatorname{Re} \left\{ \bar{c}^t A_{i\omega}^{-1} \bar{b} \right\} \geq 0$$

- (C) There is an 'effective' procedure for computing V .

The constants α, β whose existence is required are precisely those used in (3) to define V .

COMPENDIUM OF EXAMPLES

Example 1, [14] Aircraft Control System

Let us consider the following aircraft automatic control system where,

ϕ = banking angle

δ = aileron deflection

ϵ = input to the autopilot amplifier, and

T_1, K_1, K_2, K_3, K_4 = physical constants.

The rolling motion is described by

$$T_1 \ddot{\phi} + \dot{\phi} = -K_1 \delta;$$

and the input to the autopilot amplifier is given by

$$\epsilon = K_2 \phi + K_3 \dot{\phi} - K_4 \delta;$$

and the nonlinear servo is described by

$$\dot{\delta} = F(\epsilon),$$

where

$$\epsilon F(\epsilon) \geq 0.$$

By the following change of variable,

$$x_1 = \frac{1}{T_1 K_1} \dot{\phi} + \frac{1}{T_1} \delta,$$

$$x_2 = -\frac{1}{T_1} \delta,$$

$$x_3 = \frac{1}{T_1 K_1 K_3} \epsilon$$

$$= \frac{T_1 K_2}{K_3}, \quad r = \frac{K_4}{K_1 K_3}, \quad \tau = \frac{t}{T_1},$$

$$x_i = \frac{dx_i}{d\tau}, \text{ and } f(x_3) = F(\epsilon),$$

the system is transformed into the equations

$$\begin{aligned}\dot{x}_1 &= -x_1 + f(x_3), \\ \dot{x}_2 &= -f(x_3), \\ \dot{x}_3 &= (\gamma - 1)x_1 + \gamma x_2 - rf(x_3).\end{aligned}$$

The equilibrium position in this problem is not a point a phase space but is described as the condition of steady flight. That is, the aircraft will fly continuously with any small steady banking angle within some threshold zone without any action on the part of the control system. Thus, the equilibrium condition is specified in the following way:

$$\begin{aligned}x_1 = x_2 = 0, \quad |x_3| &< \frac{a}{T_1 K_1 K_3}, \text{ or} \\ \dot{\phi} = \delta = 0, \quad |\phi| &< \frac{a}{K_2}, \text{ when} \\ f(x_3) = 0 \quad \text{for } |x_3| &< \frac{a}{T_1 K_1 K_3}, \text{ and} \\ x_3 f(x_3) > 0 \quad \text{for } |x_3| &> \frac{a}{T_1 K_1 K_3}.\end{aligned}$$

In order to study the stability of the equilibrium position for $0 < \gamma < 1$, we choose the following candidate for a Liapunov function:

$$V = \frac{1 - \gamma}{2} x_1^2 + \frac{\gamma}{2} x_2^2 + \int_0^{x_3} f(x_3) dx_3,$$

where

$$\dot{V} = -(1 - \gamma) x_1^2 - r f^2(x_3).$$

If $r > 0$, \dot{V} is negative semidefinite. If $|x_3| > \frac{a}{T_1 K_1 K_3}$,

V is positive definite. Therefore, the system is asymptotically stable with respect to the equilibrium "dead-zone".

If $\delta > 1$, we choose the Liapunov function as

$$V = \frac{\delta - 1}{2} x_1^2 + \frac{\delta}{2} x_2^2 + \int_0^{x_3} f(x_3) dx_3,$$

where

$$V = -(\delta - 1) \left[f(x_3) - x_1 \right]^2 - (r + 1 - \delta) f^2(x_3).$$

Thus, \dot{V} is negative semidefinite if $r + 1 > \delta$, and V is positive definite if $|x_3| > \frac{a}{T_1 K_1 K_3}$. In summary, the system asymptotically approaches

the equilibrium "dead-zone" as $t \longrightarrow \infty$ if one of the following is true:

- (1) $0 < \delta = \frac{T_1 K_2}{K_3} < 1$ and $r = \frac{K_4}{K_1 K_3} > 0$,
- (2) $\delta = \frac{T_1 K_2}{K_3} > 1$ and $r = \frac{K_4}{K_1 K_3} > \frac{T_1 K_2}{K_3} - 1 > 0$.

Example 2, [15] Automatically Controlled Bicycle

This example deals with an automatically controlled bicycle with a nonlinear front wheel servo control. Let Θ be the angle between the bicycle frame and the vertical, x be the angle between the front wheel and a line connecting the points of contact of the wheels and the plane, and σ be the feedback signal. The nonlinear characteristic $f(\sigma)$ of the servomotor satisfies the following:

$$(1) \quad f(\sigma) = h\sigma, \quad h > 0$$

or

- (2) $f(\sigma)$ is a continuous saturation type function where $\sigma f(\sigma) > 0$, $\sigma \neq 0$.

Our problem is to establish stability conditions for the undisturbed motion of a bicycle which is controlled by a servomotor.

First, we consider the stability of the bicycle when the rolling velocity is sufficiently large. The system is described by the following equations:

$$\ddot{\Theta} = m \Theta - pW\dot{\Theta} - n x - p f(\sigma),$$

$$\dot{x} = W \dot{\Theta} + f(\sigma),$$

$$\sigma = (a + m G^2) \Theta + [E - W(N + pG^2)] \dot{\Theta} + \\ - [1/L + n G^2] x - [N + p G^2] f(\sigma),$$

where m, n, p are constants, W is the characteristic of the gyroscopic moment of the front wheel, and a, E, G^2, L and N are the control system parameters.

Let us now reduce the system equations to the normal Cauchy form by the means of $N_1 = \Theta$, $N_2 = 1/\sqrt{s} \dot{\Theta}$, $N_3 = x$, and $\tau = \sqrt{s}t$:

$$\dot{N}_1 = N_2,$$

$$\dot{N}_2 = b_{21} N_1 + b_{22} N_2 + b_{23} N_3 + h_2 f(\sigma),$$

$$\dot{N}_3 = b_{32} N_2 + f(\sigma),$$

$$\sigma = P_1 N_1 + P_2 N_2 + P_3 N_3 - (N + pG^2) f(\sigma),$$

where the new coefficients, b_{ij} , P_i , h , are related to the old coefficients through the change of variable formulas. Next, we reduce this last set of equations to canonic form. Let $\lambda_1, \lambda_2, \lambda_3$ be the roots of

$$\lambda[\lambda^2 - b_{22} \lambda - b_{21} - b_{22} b_{23}] = 0.$$

The canonic transformation is defined by

$$x_1 = \frac{b_{21}}{H(\lambda_1)} N_1 + \frac{\lambda_1}{H(\lambda_1)} N_2 + \frac{b_{23}}{H(\lambda_1)} N_3 ,$$

$$x_2 = \frac{b_{21}}{H(\lambda_2)} N_1 + \frac{\lambda_2}{H(\lambda_2)} N_2 + \frac{b_{23}}{H(\lambda_2)} N_3 ,$$

$$x_3 = - \frac{b_{32}}{h_3} N_1 + \frac{1}{h_3} N_3 ,$$

$$H(\lambda) = h_2 \lambda + b_{23} h_3 ,$$

where it is assumed that $\lambda_1 \neq \lambda_2$. In terms of the new variables, the canonic equations assume the form:

$$\dot{x}_1 = \lambda_1 x_1 + f(\sigma) ,$$

$$\dot{x}_2 = \lambda_2 x_2 + f(\sigma) ,$$

$$\dot{\chi}(\sigma) = \beta_1 x_1 + \beta_2 x_2 - R f(\sigma) ,$$

where

$$\chi(\sigma) = 1 + (N + p G^2) \frac{df}{d\sigma} ,$$

$$R = - h_2 p_2 - h_3 p_3 ,$$

$$\beta_1 = - \frac{H(\lambda_1)}{\lambda_2 - \lambda_1} \left[p_1 + \lambda_1 p_2 + b_{32} p_3 \right] ,$$

$$\beta_2 = - \frac{H(\lambda_2)}{\lambda_2 - \lambda_1} \left[p_1 + \lambda_2 p_2 + b_{32} p_3 \right] ,$$

and

$$\lambda_1 + \lambda_2 = b_{22} , \quad \lambda_1 \lambda_2 = \frac{n W - M}{S} ,$$

Let us specify the condition

$$nW - m > 0 .$$

(1)

The stability with respect to the variables x_1, x_2, x_3 also guarantees stability with respect to N_1, N_2, N_3 . Thus, the problem is to choose

the control system constants a , E , G^2 , L , N such that stability for any $f(\sigma)$ in the above class will be guaranteed.

We consider the following positive definite V-function, $R(\lambda_1) < 0$ and $R(\lambda_2) < 0$,

$$V = - \frac{a_1^2}{2\lambda_1} x_1^2 - \frac{2a_1a_2}{\lambda_1+\lambda_2} x_1 x_2 - \frac{a_2^2}{2\lambda_2} x_2^2 + \int_0^\sigma \chi(\sigma) f(\sigma) d\sigma$$

where $\chi(\sigma)$ is a nonlinear positive function of σ . The time derivative of V with reference to the canonic equations is

$$\dot{V} = - (a_1 x_1 + a_2 x_2 + \sqrt{R} f(\sigma))^2$$

if

$$\beta_1 + 2\sqrt{R} a_1 - \frac{2a_1a_2}{\lambda_1+\lambda_2} - \frac{a_1^2}{\lambda_1} = 0,$$

and

$$\beta_2 + 2\sqrt{R} a_2 - \frac{2a_1a_2}{\lambda_1+\lambda_2} - \frac{a_2^2}{\lambda_2} = 0.$$

Finally, the stability criterion is reduced to the limiting of the choice of the control system parameters by the inequalities

$$\Gamma^2 = R + \beta_1/\delta_1 + \beta_2/\delta_2 > 0, \quad R > 0,$$

$$(\lambda_1^2 + \lambda_2^2) R + \lambda_1 \beta_1 + \lambda_2 \beta_2 \pm 2\lambda_1 \lambda_2 \sqrt{R \Gamma^2} > 0,$$

and inequality (1). The above inequalities place a lower limit on the bicycle's velocity for the existence of a stable motion, along with other constraints on the control system.

The author in reference [2] also considers stability for arbitrary small velocities.

Example 3, [16] Airplane with an Autopilot

This example deals with the unperturbed motion of an airplane with an autopilot. The construction of the Liapunov functions which determine the region of permissible perturbations may be very difficult. The author simplifies the governing system of equations by a succession of transformations and then chooses a Liapunov function for the simplified system. The system is "simplified with reference to the choice of V" and thus the transformed system may not actually be written in a more concise form.

We consider the following system of equations for the perturbed motion:

$$\begin{aligned}\dot{x} &= y, \quad \dot{y} = (b_1 - b y) z^2, \\ \dot{z} &= -a x - b y - c z,\end{aligned}$$

where all the coefficients are positive constants. The first transformation is a linear substitution

$$\begin{aligned}x_1 &= x, \quad y_1 = y, \quad z_1 = \alpha x + \beta y + \gamma z, \\ \alpha &= a c, \quad \beta = b c - a, \quad \gamma = c^2\end{aligned}$$

which transforms the above system into a canonical system. The stability properties of both systems are equivalent. Another transformation is applied to the canonical system; it is defined by

$$z_1 = z, \quad \xi = x_1, \quad N = (y_1 - B x_1 z_1 - D z_1^2)(1 + A z_1),$$

where A, B, D depend on the original constants. The final form of the transformed system is given by

$$\begin{aligned}\dot{\xi} &= N/(1 + A z_1) + B \xi z_1 + D z_1^2, \\ \dot{N} &= a_1 \xi^2 + a_2 \xi N + a_4 N^2 + f_1(\xi, N, z_1), \\ \dot{z}_1 &= -c z_1 + f_2(\xi, N, z_1),\end{aligned}$$

where f_1 contains third and higher order terms in ξ, N, z_1 , and f_2 contains second and higher order terms.

Let us consider the following candidate for a Liapunov function:

$$V = N + (a_1 - a_4) \xi N - \frac{1}{2} a^2 \xi^2 - \frac{a_1}{2c} z_1^2.$$

In the neighborhood of the unperturbed motion $\xi = N = z_1 = 0$, there are points where $V > 0$ and points where $V < 0$. The time derivative of V with respect to this transformed system is

$$\dot{V} = a_1(\xi^2 + N^2 + z_1^2) + \Phi(\xi, N, z_1),$$

where Φ is third order or higher in ξ, N, z_1 .

Thus, V satisfies all the conditions of Liapunov's theorem on the instability of motion. These conditions of instability will not be satisfied if $a_1 = 0$, which is possible when $a = 0$, or when $b_1 = 0$.

The author in reference 3 then continues the stability discussion by considering the two cases: (1) $a = 0, b_1 \neq 0$, and (2) $a \neq 0, b_1 = 0$. In case (1), he shows that the unperturbed motion is unstable by considering a V-function in terms of x_1, y_1, z_1 ; namely, $V = x_1 y_1 - z_1^2 / 2c$. In case (2), he shows that the unperturbed motion is stable, but not asymptotically stable. In this case he first transforms the equations in x_1, y_1, z_1 into a new coordinate system x_2, y_2, z_2 . The Liapunov function chosen for this system is very complicated and is made up of integral terms, exponential terms and polynomial terms. From this Liapunov function, Berezkin shows that the system is stable.

Example 4, [17] Class of Nonlinear Feedback Control Systems

This concerns a study of the stability of the solutions of a third order differential equation with a discontinuous characteristic. This equation, given below, describes a definite class of nonlinear feedback control systems. In fact, it can be shown that certain problems on optimal control lead to

systems of this type. The stability of the solution is attained by increasing a parameter K (the transfer coefficient). It is shown that for a large enough value of K any operating regime of the system passes after a certain time into a "slipping" state. Hereby the dynamic error of the system becomes less than any given number.

In this report we will state the results obtained in reference [4]. Since the proof of the main theorem is very lengthy, we will not repeat it. In one part of the proof the authors of the paper use Liapunov theory, although the main part of the proof is based on other techniques.

Let us consider the differential equation

$$\ddot{x} + F(x, \dot{x}, \ddot{x}, t) + Kx \operatorname{sign} x (\dot{x} - \phi(x, \dot{x})) = 0 ,$$

where K is a positive constant. The function F is continuous in all of its argument in the region $|x| < \infty$, $|\dot{x}| < \infty$, $|\ddot{x}| < \infty$, $0 \leq t < \infty$, is bounded in t for fixed x , \dot{x} , \ddot{x} , and has continuous first order derivatives in x , \dot{x} , \ddot{x} , t . The function ϕ is continuous and has piecewise continuous first and second order derivatives with respect to x and \dot{x} in the region $|x| < \infty$ and $|\dot{x}| < \infty$.

The above third order differential equation is equivalent to the following system:

$$\begin{aligned} \dot{x} &= y , \\ \dot{y} &= z , \\ \dot{z} &= -F(x, y, z, t) - Kx \operatorname{sign} [x(z - \phi(x, y))] . \end{aligned}$$

Let us impose the following additional restrictions on the functions $\phi(x, y)$ and $F(x, y, z, t)$:

$$\begin{aligned} (a) \quad & |e^2 F(x, y/e, z/e^2, te)| < A(x, y, z) , \\ & |e\phi(x, y/e)| < B(x, y) , \end{aligned}$$

for sufficiently small values of ϵ and where A and B are continuous functions of their arguments; and

$$(b) \phi(0, 0) = 0, \quad \phi(x, 0) < 0 \quad \text{for } x \neq 0,$$

$$[\phi(x, y) - \phi(x, 0)] < 0 \quad \text{for } y \neq 0,$$

$$\int_{-\infty}^{\infty} \phi(x, 0) dx = \infty.$$

An example of F which satisfies (a) is when F is linear in x, y, z and is bounded in t for $0 \leq t < \infty$. Any linear function $\phi = cx + dy$, c and d constants, satisfies (a) and will satisfy (b) if $c < 0$ and $d < 0$.

Theorem (Proof not given here)

Let conditions (a) and (b) be satisfied, and let $\epsilon > 0$. Then, for the given bounded region G of the phase space, there exists a positive number K_0 such that for every $K \geq K_0$, any solution of the above system whose initial value lies in G will satisfy after some instant of time the condition

$$|x(t)| < \epsilon, \quad |y(t)| < \epsilon, \quad |z(t)| < \epsilon.$$

Example 5, [18] Position Control Systems

In reference [5], Fallside and Ezeilo discuss the stability of second, third and fourth order systems. The second order system is given by:

$$\dot{x} = y,$$

$$\dot{y} = -a_1 y - a_2 x - \epsilon (a_1 y + a_2 x)^2 y.$$

The candidate for a Liapunov function is the usual choice which one would make for the linearized system; namely,

$$2V = a_2 x^2 + y^2,$$

which is positive definite if $a_2 > 0$. The corresponding time derivative of V is

$$\dot{V} = -a_1 y^2 - \epsilon (a_1 y + a_2 x)^2 y^2,$$

where for $\epsilon = 0$ we get the "linearized" result. Thus, for stability (asymptotically stable in the large) we require

$$a_1 > 0, \quad a_2 > 0 \text{ and } \epsilon \geq 0.$$

The third order system is defined by

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = -a_1 z - a_2 y - a_3 x - \epsilon (a_2 y + a_3 x)^2 (z + a_1 y).$$

The Liapunov function is a quadratic form given by

$$2V = (x, y, z) \begin{pmatrix} k & k & k \\ 0 & k_2 & k_6 \\ 0 & 0 & k_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The K_1 's are chosen such that \dot{V} is defined as $\dot{V} = -f_1(x, y, z) - f_2(x, y, z)$, where f_1 and f_2 are to be perfect squares in one or more of the variables.

The resulting \dot{V} and V are

$$2V = (z + a_1 y)^2 + \frac{1}{a_2} (a_2 y + a_3 x)^2 + \\ + \frac{a_3}{a_2} (a_1 a_2 - a_3) x^2,$$

$$\dot{V} = - (a_1 a_2 - a_3) y^2 - \epsilon (a_2 y + a_3 x)^2 (z + a_1 y)^2.$$

As one can see, V is positive definite and \dot{V} is negative semidefinite it

$$a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad \epsilon \geq 0,$$

$$a_1 a_2 - a_3 > 0.$$

Therefore, these are a set of sufficient conditions for asymptotic stability in the large.

The fourth order system is

$$\dot{x} = y$$

$$\dot{y} = z$$

$$\dot{z} = w$$

$$\begin{aligned} \dot{w} = & -a_4 x - a_3 y - a_2 z - a_1 w + \\ & - \epsilon (a_3 y + a_4 x)^2 (w + a_1 z + a_2 y) . \end{aligned}$$

The quadratic form

$$2V = (x, y, z, w) \begin{pmatrix} k_1 & k_5 & k_6 & k_7 \\ 0 & k_2 & k_8 & k_9 \\ 0 & 0 & k_3 & k_{10} \\ 0 & 0 & 0 & k_4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

is taken as the Liapunov function. The K_i 's are chosen such that V has the form specified in the previous example. The results are:

$$2V = a_1 \left[w + a_1 z + (a_2 - a_3/a_1) y \right]^2 + \frac{A_4 y^2}{a_1} +$$

$$+ a_4 A_4/a_3 x^2 + a_3 \left[z + a_1 y + a_1 a_2/a_3 x \right]^2,$$

$$\begin{aligned} \dot{V} = & -A_4 y^2 - \frac{\epsilon}{a_1} \left[a_3 y + a_4 x \right]^2 \left[a_1 w + a_1^2 z + (a_1 a_2 - a_3/2)y \right]^2 \\ & + \frac{\epsilon a_3}{4 a_1} (a_3 y + a_4 x)^2 y^2 , \end{aligned}$$

where A_4 is the fourth Routh-Hurwitz condition. From V and \dot{V} the following stability conditions evolve:

$$a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, A_4 > 0,$$

$$a_1 a_2 - a_3 > 0, \epsilon \geq 0,$$

$$\epsilon (a_3 y + a_4 x)^2 < \frac{4a_1 A_4}{a_3}.$$

Example 6, [19] Nonlinear Automatic Control System

In this example a method will be given for the selection of the Θ - matrix in Malkin's method. The region of asymptotic stability obtained by this method will be compared with that obtained by Lur'e's method. It will be seen that Malkin's method gives a more conservative estimate of the region of asymptotic stability than that of Lur'e, but Malkin's method as discussed in [19] is easier to apply to an n-th order system. Below, we consider a second order system and in Example 7, we consider an n-th order system.

The canonic equations which describe the system are

$$\dot{x}_1 = \lambda_1 x_1 + f(\sigma),$$

$$\dot{x}_2 = \lambda_2 x_2 + f(\sigma),$$

$$\dot{\sigma} = \beta_1 x_1 + \beta_2 x_2 - r f(\sigma),$$

where β_1 and β_2 are characteristic constants of the system, $\lambda_1 < 0$ and $\lambda_2 < 0$ are characteristic roots of the system, $r > 0$ is the feedback coefficient, σ gives the position of the control element, and $f(\sigma)$ is the characteristic function of the control mechanism. The restrictions placed on $f(\sigma)$ are:

- (a) $f(\sigma)$ is continuous and is such that the system has a unique solution to the initial-value problem,
- (b) $f(0) = 0$, $f(\sigma) \sigma > 0$ for $\sigma \neq 0$.

The Liapunov function is of the form

$$V = \frac{1}{2} \bar{x}^t A \bar{x} + \int_0^{\sigma} f(\sigma) d\sigma$$

where A is a hermitian positive definite quadratic form. In order to make \dot{V} positive definite it is necessary to select A such that the matrix defined by

$$\Theta = \frac{1}{2} (A \lambda + \lambda^t A) < \Theta$$

and also such that the matrix

$$\begin{pmatrix} -\Theta & \bar{g} \\ \bar{g}^t & r \end{pmatrix} ; \quad 2\bar{g} = -\bar{\beta} - A \bar{e}$$

is positive definite. With Θ negative definite this last requirement adds only that

$$\Delta = \begin{vmatrix} -\Theta & \bar{g} \\ \bar{g}^t & r \end{vmatrix} > 0$$

Choose the Θ - matrix in Malkin's method to be of the form:

$$\Theta = \begin{pmatrix} -\epsilon_1 & 0 \\ 0 & -\epsilon_2 \end{pmatrix},$$

where ϵ_1 and ϵ_2 are positive. Matrix A is a symmetric, positive definite matrix defined by

$$\Theta = \frac{1}{2} [A \lambda + \lambda^t A],$$

where

$$\lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

thus, matrix A becomes,

$$A = \begin{pmatrix} -\epsilon_1/\lambda_1 & 0 \\ 0 & -\epsilon_2/\lambda_2 \end{pmatrix}.$$

The column matrix \underline{g} is defined by:

$$\underline{g} = - \frac{1}{2} \left\{ \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \underline{A} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \frac{1}{2} \begin{pmatrix} 1/\lambda_1 - 1 \\ 2/\lambda_2 - 2 \end{pmatrix}.$$

If the following condition is fulfilled, the equilibrium solution is asymptotically stable:

$$\Delta = \begin{vmatrix} -\Theta \\ \underline{g}^t \end{vmatrix} > 0,$$

or

$$\frac{1}{\epsilon_1} \left(\frac{\epsilon_1}{2\lambda_1} - \frac{\beta_1}{2} \right)^2 + \frac{1}{\epsilon_2} \left(\frac{\epsilon_2}{2\lambda_2} - \frac{\beta_2}{2} \right)^2 < r.$$

The constants ϵ_1 and ϵ_2 are arbitrary, thus the above inequality will give a maximum stability region for a given form of Θ . In [19], the following region is obtained:

$$\beta_1 < -r\lambda_1$$

$$\beta_2 < -r\lambda_2 \quad (\beta_1 < 0)$$

$$\beta_2 < -r\lambda_2 - \frac{\lambda_2}{\lambda_1} \beta_1 \quad (\beta_1 \geq 0).$$

Also, in [19], this region is compared with results obtained by Lur'e.

Example 7, [19] N-th Order Case of Example 6

In this example we consider the n th order case corresponding to Example 6.

The canonical equations are

$$\begin{aligned} \dot{\underline{x}} &= \underline{\lambda} \underline{x} + f(\sigma) \underline{e}, \\ \dot{\sigma} &= \underline{\beta}^t \underline{x} - r f(\sigma), \end{aligned}$$

where \underline{x} , $\underline{\beta}$, \underline{e} are column matrices of n -th order containing elements x_i , β_i , $e_i = 1$, respectively; $\underline{\lambda}$ is a diagonal matrix with elements λ_i . The characteristic roots,

λ_i , are real, negative and all different.

Then (as in Example 6) we take the $\underline{\Theta}$ - matrix to be of the form

$$\underline{\Theta} = \begin{pmatrix} -\epsilon_1 & 0 & \dots & 0 \\ 0 & -\epsilon_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\epsilon_n \end{pmatrix}$$

where $\epsilon_i > 0$ are arbitrary. The \underline{A} and \underline{g} matrices used in Malkin's method are defined by

$$a_{ii} = -\epsilon_i / \lambda_i, \quad a_{ik} = 0 \quad (i \neq k), \quad g_i = \frac{\epsilon_i}{2\lambda_i} - \frac{\beta_i}{2}.$$

A sufficient condition for asymptotic stability of the equilibrium position is that

$$\Delta = \begin{vmatrix} -\underline{\Theta} & \underline{g} \\ \underline{g}^t & r \end{vmatrix} > 0.$$

Expanding this determinant gives the following inequality:

$$\frac{1}{\epsilon_1} \left(\frac{\epsilon_1}{2\lambda_1} - \frac{\beta_1}{2} \right)^2 + \dots + \frac{1}{\epsilon_n} \left(\frac{\epsilon_n}{2\lambda_n} - \frac{\beta_n}{2} \right)^2 < r.$$

The maximum region of asymptotic stability given by this inequality is

$$\beta_i < -\lambda_i r, \quad \begin{cases} \beta_k < -\lambda_k r & (\beta_i < 0) \\ \beta_k < -\lambda_k (r - S_k) & (\beta_i \geq 0) \end{cases},$$

where $k = 1, \dots, n$; $i = 1, \dots, k-1, k+1, \dots, n$; $S_k = \sum_{i=1}^k \frac{\beta_i}{\lambda_i}$.

These conditions apply to the case of real negative roots.

Example 8, [19] Nth Order Control System

In this example Komarnitskaia considers the system whose characteristic equation has complex conjugate roots with negative real parts as well as real negative roots.

Let λ_i, x_i, β_i ($i = 1, \dots, 2S$) be complex conjugate pairs, λ_j, x_j, β_j ($j = 2S + 1, \dots, n$) be real numbers; also, let $\lambda_j < 0$ and $\operatorname{Re}(\lambda_i) < 0$.

In this case Malkin's Θ - matrix takes the form

$$\Theta = \begin{bmatrix} 0 & -\epsilon_1 & \dots & 0 & 0 & 0 & \dots & 0 \\ -\epsilon_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 & -\epsilon_{2S-1} & \cdot & \dots & \cdot \\ 0 & 0 & \dots & -\epsilon_{2S-1} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & -2\epsilon_{2S+1} & \dots & 0 \\ \cdot & \cdot & \dots & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & -2\epsilon_n \end{bmatrix}$$

where $\epsilon_i > 0$ are arbitrary. The same Liapunov function is used as in the previous examples and its time derivative is

$$\dot{V} = \bar{x}^t \Theta \bar{x} + (A \bar{e} + \bar{A})^t \bar{x} f(\sigma) - r f^2(\sigma)$$

The elements of the A matrix are

$$a_{ii} = 0, a_{ii+1} = -\frac{\epsilon_i}{\lambda_i + \lambda_{i+1}} \quad i = 1, 3, 5, \dots, 2S-1$$

$$a_{ii} = -\frac{\epsilon_i}{\lambda_i} \quad i = 2S + 1, \dots, n$$

all other $a_{ik} = 0$.

We shall write

$$x_k = u_k + u_{k+1} i, \beta_k = \delta_k + \delta_{k+1} i$$

$$x_{k+1} = u_k - u_{k+1} i, \beta_{k+1} = \delta_k - \delta_{k+1} i \quad k=1, 3, \dots, 2S-1$$

Then we have

$$\begin{aligned}
 \dot{V} = & - \sum_{k=1,3}^{2S-1} \epsilon_k (u_k^2 + u_{k+1}^2) - \sum_{k=2S+1}^n \epsilon_k x_k^2 \\
 & + \sum_{k=2S+1}^n \left(\beta_k - \frac{\epsilon_k}{\lambda_k} \right) x_k f(\sigma) \\
 & + 2 \sum_{k=1}^s \left[\left(\gamma_{2k-1} - \frac{\epsilon_{2k-1}}{\lambda_{2k-1} + \lambda_{2k}} \right) u_{2k-1} + \gamma_{2k} u_{2k} \right] f(\sigma) \\
 & - r f^2(\sigma)
 \end{aligned}$$

The condition for absolute stability becomes

$$\begin{vmatrix}
 \epsilon_1 & 0 & 0 & \dots & 0 & \delta_1^* \\
 0 & \epsilon_1 & 0 & \dots & 0 & \delta_2^* \\
 0 & 0 & \epsilon_3 & \dots & 0 & \delta_3^* \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \dots & n & \epsilon_n \delta_n^* \\
 \delta_1^* & \delta_2^* & \delta_3^* & \dots & B_n^* & r
 \end{vmatrix} > 0$$

where $\delta_1^* = \frac{\epsilon_1}{\lambda_1 + \lambda_2} - \delta_1$, $\delta_2^* = -\delta_2$

$$\delta_3^* = \frac{\epsilon_3}{\lambda_3 + \lambda_4} - \delta_3, \dots, \delta_n^* = \frac{\epsilon_n}{2\lambda_n} - \frac{\beta_n}{2}$$

or

$$\begin{aligned}
 r > & \frac{1}{\epsilon_1} \left(\frac{\epsilon_1}{\lambda_1 + \lambda_2} - \delta_1 \right)^2 + \frac{\delta_2^2}{\epsilon_1} + \dots + \frac{1}{\epsilon_{2s-1}} \left(\frac{\epsilon_{2s-1}}{\lambda_{2s-1} + \lambda_{2s}} - \delta_{2s-1} \right)^2 \\
 & + \frac{\delta_{2s}^2}{\epsilon_{2s-1}} + \dots + \frac{1}{\epsilon_n} \left(\frac{\epsilon_n}{2\lambda_n} - \frac{\beta_n}{2} \right)
 \end{aligned}$$

and using the arbitrariness of ϵ_1, ϵ_2 the conditions for absolute stability become

$$\beta_i < -\frac{m_i \lambda_i}{2}, \quad \left\{ \begin{array}{l} \delta_{j+1}^2 < l_j (\lambda_j + \lambda_{j+1}) \delta_j + \frac{1}{4} (\lambda_j + \lambda_{j+1})^2 l_j^2 \\ \delta_{2k+2}^2 < (r - \sum_j l_j) (\lambda_{2k+1} + \lambda_{2k+2}) \delta_{2k+1} \\ \quad + \frac{1}{4} (\lambda_{2k+1} + \lambda_{2k+2})^2 (r - \sum_j l_j)^2 \\ \delta_{j+1}^2 < l_j (\lambda_j + \lambda_{j+1}) \delta_j + \frac{1}{4} (\lambda_j + \lambda_{j+1})^2 l_j^2 \\ \delta_{2k+2}^2 < (r - \sum_j l_j - \sum_i m_i) (\lambda_{2k+1} + \lambda_{2k+2}) \delta_{2k+1} \\ \quad + \frac{1}{4} (\lambda_{2k+1} + \lambda_{2k+2})^2 (r - \sum_j l_j - \sum_i m_i)^2 \end{array} \right. \quad \begin{array}{l} \beta_i < 0 \\ \beta_i \geq 0 \end{array}$$

$i = 2s+1, \dots, n; j=1,3,\dots,2k-1,2k+3,\dots,2s-1; k=1,\dots,s-1$

where $l_j = \frac{-2}{\lambda_j + \lambda_{j+1}} (\delta_j + \sqrt{\delta_j^2 + \delta_{2j+1}^2})$, $m_i = -\frac{2\beta_i}{\lambda_i}$

are positive numbers such that

$$\sum_i m_i + \sum_j l_j \leq r$$

As an example consider a system described by

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + f(\sigma) & \lambda_1 &= M_1 + M_2 i \\ \dot{x}_2 &= \lambda_2 x_2 + f(\sigma) & \lambda_2 &= M_1 - M_2 i \\ \dot{\sigma} &= \beta_1 x_1 + \beta_2 x_2 - r f(\sigma) & \beta_1 &= \delta_1 + \delta_2 i \\ & & \beta_2 &= \delta_1 - \delta_2 i \end{aligned}$$

with $M_1 < 0$, M_2 and $\delta_2 \neq 0$, and for all other quantities the same assumptions as in the general case apply. The stability region is defined by the inequality.

$$\delta_2^2 < r(\lambda_1 + \lambda_2) \delta_1 + \frac{1}{4} (\lambda_1 + \lambda_2)^2 r^2$$

The stability region is shown in fig. (1)

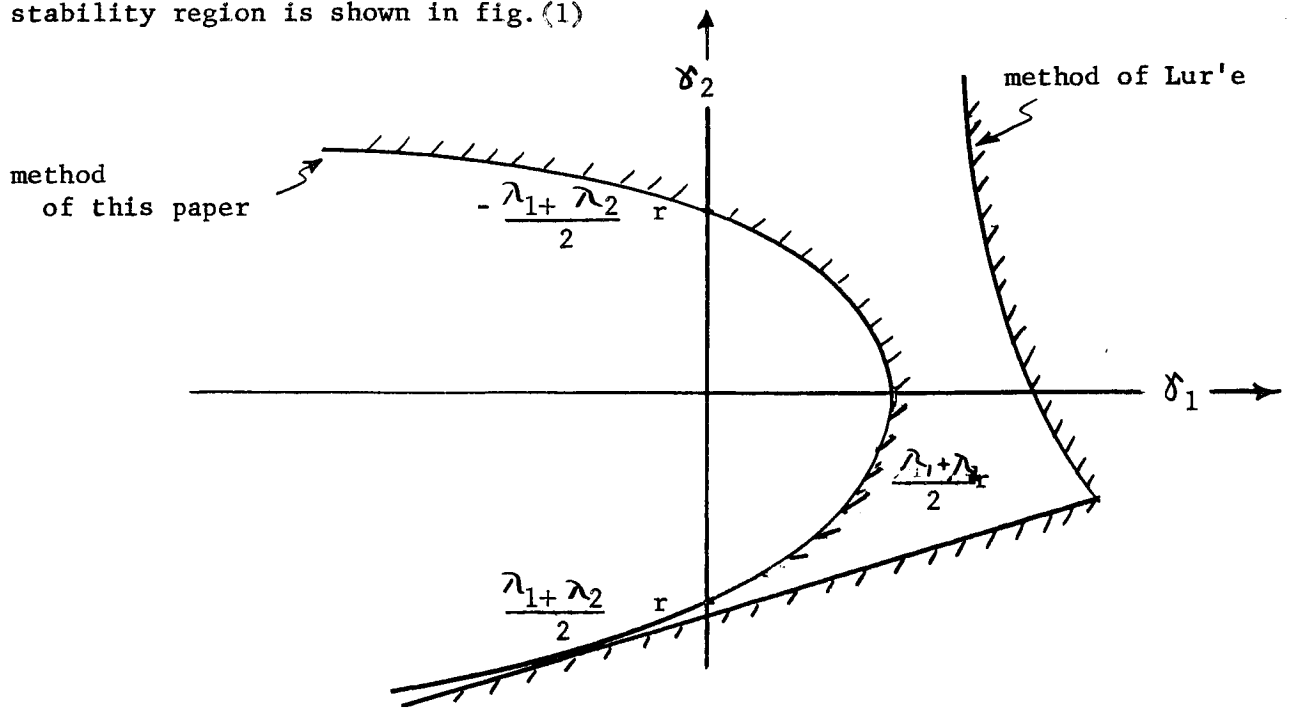


fig. (1)

Example 9 [20]

The equation for a hydraulic actuator, considering the load, is

$$\ddot{\xi} = \frac{f(\sigma)}{\sqrt{\frac{g p_0}{\delta}}} \frac{Q(w)}{\frac{Q}{F}} \quad (1)$$

where $f(\sigma) = \sqrt{\frac{g p_0}{\delta}} \frac{Q}{F} u \sigma$ is the velocity of the unloaded actuator and $Q(w)$ is the effect of the load. Here $\sqrt{\frac{g p_0}{\delta}} \frac{Q}{F}$ is a constant, u is the pump discharge coefficient, σ is the valve displacement and p_0 is the difference between the pressure in the supply line and the return line. The effect of the load, $Q(w)$, is given by

$$\begin{aligned} &1 \text{ for } w \geq 1 \\ &Q(w) = \sqrt{w} \text{ for } 0 < w < 1 \\ &0 \text{ for } w \leq 0 \end{aligned} \quad (2)$$

where w is, in general, of the form

$$W = 1 - (a \ddot{\xi} + b \dot{\xi} + c \xi) \text{sign} \sigma \quad (3)$$

(sign σ is the Kroneher function = -1 if $\sigma < 0$, 0 if $\sigma = 0$, +1 if $\sigma > 0$)

We consider three special cases of (3), viz.

$$w = 1 - c \dot{\xi} \operatorname{sign} \sigma \quad (4)$$

$$w = 1 - b \dot{\xi} \operatorname{sign} \sigma \quad (5)$$

$$w = 1 - (b \dot{\xi} + c \ddot{\xi}) \operatorname{sign} \sigma \quad (6)$$

For each of these cases we reduce the equation (1) to Cauchy normal form, obtaining

$$\ddot{\xi} = f(\sigma) Q_1(\sigma) \quad (7)$$

$$\ddot{\xi} = f(\sigma) Q_2(\sigma) \quad (8)$$

$$\ddot{\xi} = f(\sigma) Q_3(\sigma) \quad (9)$$

respectively, where

$$Q_1(\sigma) = \sqrt{1 - c \dot{\xi} \operatorname{sign} \sigma}$$

$$Q_2(\sigma) = \frac{1}{2} \left[-b f(\sigma) \operatorname{sign} \sigma + \sqrt{b^2 f^2(\sigma) + 4} \right]$$

$$Q_3(\sigma) = \frac{1}{2} \left[-b f(\sigma) \operatorname{sign} \sigma + \sqrt{b^2 f^2(\sigma) + 4(1 - c \dot{\xi} \operatorname{sign} \sigma)} \right]$$

Notice that (9) is meaningful only if $w > 0$.

This will be so if $1 - \dot{\xi} c \operatorname{sign} \sigma > 0$ is satisfied. If (10) is not satisfied, the actuator is not operating and $\dot{\xi} = 0$. (10)

Consider the indirect control system

$$\begin{aligned} \dot{N}_d &= \sum_k b_{dk} N_k + n_d \quad (d = 1, \dots, n) \\ \dot{\xi} &= f(\sigma) Q(w) \quad (k = 1, \dots, n) \end{aligned} \quad (11)$$

$$\sigma = \sum_k p_k N_k - r \dot{\xi}$$

where b_k , p_k , n , and r are constants. We write (11) in canonical form as follows:

$$\begin{aligned} \dot{x}_d &= -p_d x_d + f(\sigma) Q(w) \quad (d = 1, \dots, n) \\ \sigma &= \sum_k c_k x_k - r \dot{\xi} \quad (k = 1, \dots, n) \end{aligned} \quad (12)$$

$$\dot{\sigma} = \sum_d B_d x_d - r f(\sigma) Q(w)$$

where c_k , q , r , B_d , and P_d are constants.

Notice that if $\dot{\xi} = 0$, then

$$\dot{x}_d = -P_d x_d \quad (13)$$

$$\sigma = \sum_k c_k x_k - q \xi$$

The problem is to choose the B_k such that the equilibrium point

$$x_i = \sigma = \xi = 0$$

is stable. We choose as a Liapunov function

$$V = \sum_i \sum_k \frac{a_i a_k}{P_i + P_k} x_i x_k + \frac{1}{2} \sum_k A_k x_k^2 + \int_0^\sigma f(\sigma) Q(\sigma) d\sigma \quad (14)$$

Taking the total derivative w.r.t.t,

$$\begin{aligned} \dot{V} = & - \left[\sum_k a_k x_k + \sqrt{r} f(\sigma) Q(\sigma) \right]^2 - \sum_k p_k A_k x_k^2 \\ & + \left[\int_0^\sigma \frac{\partial Q}{\partial \xi} f(\sigma) d\sigma \right] f(\sigma) Q(\sigma) \end{aligned} \quad (15)$$

as long as

$$A_k + 2 \sqrt{r} a_k + 2 a_k \sum_i \frac{a_i}{P_i + P_k} + B_k = 0 (k=1, \dots, n) \quad (16)$$

is satisfied. Notice that the conditions placed on the system along with the additional restriction

$$f(0) = 0, f(\sigma)\sigma > 0 \quad \text{if} \quad \sigma \neq 0$$

insure that (14) is positive definite. Furthermore, for the various choices

(7), (8), (9) for $Q(\sigma)$, (15) is always negative in the region in which the

actuator is operating and $\sigma \neq 0$. If $\sigma > 0$, $\partial Q_1 / \partial \xi < 0$ and $\partial Q_3 / \partial \xi < 0$.

If $\sigma < 0$, $\partial Q_1 / \partial \xi > 0$ and $\partial Q_3 / \partial \xi > 0$. Of course $\frac{\partial Q_2}{\partial \xi} = 0$. Hence the system is stable within the operating range of the actuator whenever (16) is satisfied.

If the actuator is not operating, $\dot{\xi} = 0$, then, as can be seen from (13), the x_i are attenuated and tend toward the origin. Hence the system (11) is stable for all three choices of $Q(\sigma)$, as long as (16) is satisfied.

Example 10 [21]

Consider the third-order indirect-control system

$$\begin{aligned}\dot{z}_p &= \lambda_p z_p + f(\sigma) \quad (p = 1, 2) \\ \dot{\sigma} &= B_1 z_1 + B_2 z_2 - r f(\sigma)\end{aligned}\quad (1)$$

where it is assumed that $\operatorname{Re} \lambda_p < 0$, $r > 0$, $f(0) = 0$, $0 < \frac{f(\sigma)}{\sigma} < \infty$ ($\sigma \neq 0$), and that $f(\sigma)$ is defined for all real σ .

The Routh-Hurwitz theorem will be used to establish necessary and sufficient conditions for the absolute stability of (1). These conditions will be compared with the sufficient conditions obtained from the methods of Popov and Lur'e, and will be applied to the second problem of Bulgakov.

Assume that (1) is absolutely stable. Then the linear system obtained by letting $f(\sigma) = h \sigma$ is asymptotically stable. By the Routh-Hurwitz theorem, the following inequalities hold:

$$\begin{aligned}a_1 + r h &> 0 \\ b_2 h &> 0 \\ A h^2 + B h + c &> 0\end{aligned}\quad (2)$$

where $A = r b_1$, $B = r a_2 + a_1 b_1 - b_2$, $C = a_1 a_2$

$$b_1 = a_1 r - (B_1 + B_2) \quad b_2 = r a_2 + B_1 \lambda_2 + B_2 \lambda_1$$

$$a_1 = -(\lambda_1 + \lambda_2) \quad a_2 = \lambda_1 \lambda_2$$

In light of the conditions placed on the system (1), the conditions (2) are

$$\begin{aligned}\text{equivalent to either } A > 0 \text{ and } -2 \sqrt{AC} < B < \frac{r^2 a_2 + a_1 A}{r} \\ \text{or} \quad A = 0 \text{ and } 0 \leq B < r a_2\end{aligned}\quad (3)$$

The conditions obtained by the method of Lur'e for absolute stability are

$$T^2 = \frac{B_1}{\lambda_1} + \frac{B_2}{\lambda_2} + r > 0, \text{ and}$$

$$D^2 = r(\lambda_1^2 + \lambda_2^2) + \lambda_1 B_1 + \lambda_2 B_2 + 2 \lambda_1 \lambda_2 \sqrt{r} T > 0 \text{ and } r \geq 0.$$

For the system under consideration, these conditions are equivalent to: either

$$r^2 a_2 + a_1 A - B r > 0 \text{ and } B \geq 2 r a_2 \quad (4)$$

$$\text{or } B^2 - 4 A C < 0.$$

These conditions are considerably more restrictive than (3), as is easily seen by examining the regions they determine in the A, B plane. Hence, the method of Lur'e does not yield necessary conditions for absolute stability.

V. M. Popov (in A. + R.C. V19#1) extended the region given by (4) to the following,

$$A > 0 \text{ and } -2 \sqrt{AC} < B < \frac{r^2 a_2 + a_1 A}{r} \quad (5)$$

but did not consider the two cases

$$A = 0 \text{ and } 0 < B < r a_2 \quad (6)$$

$$\text{and } A = 0 \text{ and } B = 0. \quad (7)$$

V. A. Pliss studied case (6) and proved that in that case the system (1) is absolutely stable. However it can be shown that the condition (7) does not satisfy Popov's criterion. Hence, the method of Popov does not yield necessary conditions for absolute stability, although in this case the conditions are much broader than those of Lur'e.

In the second method of Bulgakov we consider the system

$$T^2 \ddot{\psi} + u \dot{\psi} + K \psi + \mu = 0 \quad (8)$$

$$\dot{\mu} = f^*(\sigma), \sigma = a \psi + E \dot{\psi} + G^2 \ddot{\psi} - \frac{\mu}{L}$$

Changing notation we let

$$\psi = N_1, \dot{\psi} = \sqrt{\delta} N_2, \mu = i \xi, t = \frac{\tau}{\sqrt{\delta}},$$

$$\bar{p} = \frac{\mu}{T^2}, n_2 = -1, i = \frac{L T^2}{T^2 + L G^2}, f(\sigma) = \frac{1}{\sqrt{\delta}} f^*(\sigma)$$

$$b_{21} = -\frac{q}{r}, b_{22} = \frac{\bar{p}}{\sqrt{\delta}}, p_1 = a - \bar{q} c^2, p_2 = (E - \bar{p} G^2) \sqrt{\delta}$$

The system (8) is then reduced to the normal form

$$\begin{aligned} \dot{N}_1 &= N_2, \quad \dot{N}_2 = b_{21}N_1 + b_{22}N_2 + n_2\xi, \\ \dot{\xi} &= f(\sigma), \quad \sigma = p_1 N_1 + p_2 N_2 - \xi \end{aligned} \quad (9)$$

A. M. Letov studied this system and found the region of absolute stability to be the region inside the parabola $(N + \xi - 1)^2 = 4/N\xi$. The criteria (3) extend this region to the entire quadrant $\xi > 0$, $N \geq 0$ except for the curvilinear triangle formed by the segments $[0, 1]$ of the ξ and N axes and the parabola $(N + \xi - 1)^2 = 4 N \xi$. The only part of the boundary included in the region is the half line $N = 0$, $\xi \geq 1$.

It should be noted in passing that the Aizerman conjecture has an affirmative answer for the system (1), i.e., the region of absolute stability for the system (1) coincides with the stability region when $f(\sigma) = h\sigma$.

Example 11, Letov [5], Chang [22]

Letov considered an automatic control system described by the canonical equations

$$\begin{aligned} \dot{\bar{x}} &= A \bar{x} - \bar{b} f(\sigma) \\ \dot{\sigma} &= \bar{c}^t \bar{x} - \rho f(\sigma) \end{aligned} \quad (1)$$

where $A = \begin{pmatrix} -p_1 & 0 & 0 & 0 \\ 0 & -p_2 & 0 & 0 \\ . & . & . & . \\ . & . & 0 & -p_n \end{pmatrix}$, $\bar{b} = \begin{pmatrix} -1 \\ -1 \\ . \\ . \\ -1 \end{pmatrix}$

and the p_i are distinct and real numbers greater than zero.

As a candidate for a Liapunov function he chose

$$V = \frac{1}{2} \bar{x}^t \bar{x} + R \int_0^\sigma f(\sigma) d\sigma \quad (2)$$

which upon differentiation with respect to time yields

$$\begin{aligned}
 \dot{\bar{V}} &= \frac{1}{2} \frac{\bar{x}^t}{\bar{x}} \bar{x} + \frac{1}{2} \frac{\bar{x}^t}{\bar{x}} \bar{x} + R f(\sigma) \dot{\sigma} \\
 &= \frac{1}{2} (\bar{x}^t A^t \bar{x} + \bar{x}^t A \bar{x}) + R f(\sigma) (\bar{c}^t \bar{x} - \rho f(\sigma)) \\
 &\quad - \frac{1}{2} f(\sigma) \bar{b}^t \bar{x} - \frac{1}{2} f(\sigma) \bar{x}^t \bar{b} \\
 &= \bar{x}^t A \bar{x} - R \rho f(\sigma)^2 + f(\sigma) (-\bar{b}^t + R \bar{c}^t) \bar{x} \\
 \dot{\bar{V}} &= (\bar{x}, f) \begin{pmatrix} -A & \frac{1}{2} (\bar{b} - R \bar{c}) \\ \frac{1}{2} (\bar{b} - R \bar{c})^t & R \rho \end{pmatrix} \begin{pmatrix} \bar{x} \\ f \end{pmatrix} \quad (3)
 \end{aligned}$$

Since $-A$ is positive definite we need only make

$$\begin{vmatrix} -A & \frac{1}{2} (\bar{b} - R \bar{c}) \\ \frac{1}{2} (\bar{b} - R \bar{c})^t & R \rho \end{vmatrix} > 0.$$

to have $\dot{\bar{V}}$ negative definite. Hence Letov obtained the stability condition

$$R \rho > \frac{1}{4} (\bar{b} - R \bar{c})^t (-A)^{-1} (\bar{b} - R \bar{c}) \quad (4)$$

Chang points out that this can be written

$$R^2 \left(\bar{c}^t A^{-1} \bar{c} \right) - 2 R \left[\bar{b}^t A^{-1} \bar{c} - 2 \rho \right] + \bar{b}^t A^{-1} \bar{b} > 0 \quad (5)$$

or

$$-R^2 \ell - R m - s > 0 \quad (6)$$

where $\ell = -\bar{c}^t A^{-1} \bar{c}$, $m = 2 \bar{b}^t A^{-1} \bar{c} - 2\rho$, $s = -\bar{b}^t A^{-1} \bar{b}$

Let M be a constant and introduce the variables x, y

$$y - x = \frac{m}{\ell}, \quad x - y = -\frac{s}{\ell} + M \quad (7)$$

x and y are variables in the sense that we treat the parameters of the system, reflected here in m, ℓ, s , as variables. Then (6) becomes

$$(x - R)(y + R) > M \quad (8)$$

In the case of equality in (8) we have

$$(x - R)(y + R) = M \quad (9)$$

which is an equation of a hyperbola. For each value of R there is a hyperbola in the x, y plane which bounds a region of stability. Chang shows that the region of stability as given by (8) is on the concave side of the hyperbola given by (9). Then the envelope of the family of hyperbolas is the envelope of the stability region in the parameter space (x, y -plane). This envelope can be found as follows. Let

$$F(x, y, R) = (x-R)(y+R) - M = 0$$

Then the envelope is found by eliminating R between

$$F(x, y, R) = 0, \quad \frac{\partial}{\partial R} F(x, y, R) = 0$$

This yields the envelope (two straight lines)

$$(x + y)^2 - 4M = 0$$

and the value of R

$$R = \frac{1}{2} (x-y) = -\frac{1}{2} \frac{m}{\ell}$$

$$R = \frac{2\rho - \frac{t}{b} \frac{-1}{A} \frac{1}{c}}{\frac{-t}{c} \frac{-1}{A} \frac{1}{c}} \quad (10)$$

Upon substituting (10) in (5) Chang obtains the single criterion

$$\frac{2\rho - \frac{t}{b} \frac{-1}{A} \frac{1}{c}}{\frac{-t}{c} \frac{-1}{A} \frac{1}{c}} > -\frac{t}{b} \frac{-1}{A} \frac{1}{b} \quad (11)$$

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